Journal of Mathematical Research & Exposition
Vol.13, No.3, Aug. 1993

A Theorem on the Steepest Descent Direction for Linearly Constrained Problem*

Zhang Liwei *Feng Enmin* *Xia Zunquan*
(Dept. of Appl. Math., Dalian Univ. of Tech., Dalian)

In this short note the relationship between the steepest descent direction and ABS descent directions for linearly constrained minimization with equality constraints is given. Consider the problem (LEP): $\min\{f(x) \mid Ax = b, x \in \mathbf{R}^n\}$, where $A \in \mathbf{R}^{m,n}$, A is of full rank in row. Let H denote the last ABS update matrix of an ABS algorithm to A, i.e., H_{m+1}, especially, H^{Huang} is used instead of H in the case where the Huang algorithm is adopted and $S(x)$ the feasible direction set of (LEP) at x when x is a feasible point. In the ABS form one has $S(x) = \{d \in \mathbf{R}^n \mid d = H^T q, q \in \mathbf{R}^n\}$. If x is not a local minimizer then the steepest descent direction problem for (LEP) is of the one $\min\{g_x^T d \mid Ad = 0, \|d\|^2 = c\}$, where c is a positive scalar.

Lemma *If $Hg \neq 0$ and $r(A) = m$ then the direction $d = -H^{Huang} g_x / \|H^{Huang} g_x\|$ is the steepest descent direction for (LEP) disregarding a constant.*

Theorem *Let x be a feasible point. Suppose that $Hg_x \neq 0$ where $g_x = \nabla f(x)$. Then we have the following assertions: (i) there exists a $q^* \in \mathbf{R}^n$ such that $H^T q^*$ is the steepest descent direction and equal to $-H^{Huang} g_x / \|H^{Huang} g_x\|$; (ii) the general solution of q in (i) can be formulated in the form $q = q^* + W^m \eta$, $\eta \in \mathbf{R}^m$, where W is an ABS parameter matrix; (iii) for any H one has $g_x^T(-H^T H g_x / \|H^T H g_x\| + H^{Huang} g_x / \|H^{Huang} g_x\|) = \|g_x\| \cos\theta_0 (1 - \cos\theta)$, where θ_0 is the angle between g_x and $-H^{Huang} g_x$, and θ the one between $-H^T H g_x$ and $-H^{Huang} g_x$.*

Proof The proof of (i). The problem $\min\{g_x^T H^T q \mid \|H^T q\|^2 = c\}$ is the steepest descent one to (LEP) where c is a positive scalar. According to the K-T conditions there exists a scalar μ such that $\{Hg_x + \mu H H^T q = 0, q^T H H^T q = c\}$. Since $g_x + \mu H^T q \in \mathrm{Null}(H)$, there exists a $\nu \in \mathbf{R}^m$ such that $g_x + \mu H^T q = A^T \nu$ and hence $\nu = A^{+T} g_x$. Furthermore, one has that $H^T q = -(I - A^+ A) g_x / \mu = -H^{Huang} g_x / \mu$. It follows from the second equation of the above system that $\mu = \|H^{Huang} g_x\| / c^{1/2}$ and there exists at least one $q*$ such that $H^T q* = -c^{1/2} H^{Huang} g_x / \|H^{Huang} g_x\|$ is the steepest descent direction to (LEP). The fact that $-H^{Huang} g_x / \|H^{Huang} g_x\|$ is the steepest descent direction can be proved by the similar way. The scalar c can be taken as one. The proof of (ii). It can be done by properties of the ABS algorithm. The proof of (iii). It can be proved in terms of the facts that $g_x^T H^T H g_x / \|H^T H g_x\| = \|g_x\| \cos\theta_1$, $g_x^T H^{Huang} g_x / \|H^{Huang} g_x\| = \|g_x\| \cos\theta_0$ and $\cos\theta_1 = \cos\theta_0 \cos\theta$, where we denote by θ_1 the angle between $-g_x$ and $H^T H g_x$. □

*Received Jan. 9, 1993.

© 1995-2005 Tsinghua Tongfang Optical Disc Co., Ltd. All rights reserved.