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s-REGULAR DIHEDRAL COVERINGS OF THE COMPLETE GRAPH OF ORDER 4***

FENG Yanquan*   J. H. KWAK**

Abstract

A graph is s-regular if its automorphism group acts regularly on the set of its s-arcs. An infinite family of cubic 1-regular graphs was constructed in [7] as cyclic coverings of the three-dimensional Hypercube, and a classification of all s-regular cyclic coverings of the complete bipartite graph of order 6 was given in [8] for each $s \geq 1$, whose fibre-preserving automorphism subgroups act arc-transitively. In this paper, the authors classify all s-regular dihedral coverings of the complete graph of order 4 for each $s \geq 1$, whose fibre-preserving automorphism subgroups act arc-transitively. As a result, a new infinite family of cubic 1-regular graphs is constructed.

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§ 1. Introduction

For a finite, simple, and undirected graph X, every edge of X gives rise to a pair of opposite arcs, and we denote by $V(X)$, $E(X)$, $A(X)$ and $\mathrm{Aut}(X)$ the vertex set, the edge set, the arc set and the automorphism group of X, respectively. The neighborhood of a vertex $v \in V(X)$, denoted by $N(v)$, is the set of vertices adjacent to v. Let a group G act on a set Ω, and let $\alpha \in \Omega$. We denote by G_α the stabilizer of α in G, that is, the subgroup of G fixing α. The group G is said to be semiregular if $G_\alpha = 1$ for each $\alpha \in \Omega$, and regular if G is semiregular and transitive on Ω. A graph $\widetilde{X}$ is called a covering of X with a projection $p : \widetilde{X} \to X$, if p is a surjection from $V(\widetilde{X})$ to $V(X)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(X)$ and $\tilde{v} \in p^{-1}(v)$. The graph $\widetilde{X}$ is also called the covering graph and X is the base graph. A covering $\widetilde{X}$ of X with a projection p is said to be regular (or K-covering) if there is a semiregular subgroup K of the automorphism group $\mathrm{Aut}(\widetilde{X})$ such that the graph X is isomorphic to the quotient graph $\widetilde{X}/K$, say by h, and the quotient map $\widetilde{X} \to \widetilde{X}/K$ is the composition ph of p and h (for the purpose of this paper, all functions

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*Department of Mathematics, Northern Jiaotong University, Beijing 100044, China.
E-mail: yqfeng@center.njtu.edu.cn
Combinatorial and Computational Mathematics Center, Pohang University of Science and Technology, Pohang, 790–784, Korea. **E-mail: jinkwak@postech.ac.kr
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are composed from left to right). If K is cyclic, elementary abelian or dihedral then $\widetilde{X}$ is called a cyclic, an elementary abelian or a dihedral covering of X respectively, and if $\widetilde{X}$ is connected then K is called the covering transformation group. The fibre of an edge or a vertex is its preimage under p. An automorphism of $\widetilde{X}$ is said to be fibre-preserving if it maps a fibre to a fibre, while an element of the covering transformation group fixes each fibre setwise.

An s-arc in a graph X is an ordered $(s+1)$-tuple $(v_0, v_1, \ldots, v_{s-1}, v_s)$ of vertices of X such that v_{i-1} is adjacent to v_i for $1 \le i \le s$ and $v_{i-1} \ne v_{i+1}$ for $1 \le i < s$. A graph X is said to be s-arc-transitive if $\mathrm{Aut}(X)$ is transitive on the set of s-arcs in X. In particular, 0-arc-transitive means vertex-transitive, and 1-arc-transitive means arc-transitive or symmetric. A symmetric graph X is said to be s-regular if the automorphism group $\mathrm{Aut}(X)$ acts regularly on the set of s-arcs in X. Tutte [22, 23] showed that every finite cubic symmetric graph is s-regular for some s, and this s should be at most five. A subgroup of the automorphism group of a graph is said to be s-regular if it acts regularly on the set of s-arcs in the graph.

Djoković and Miller [5] constructed an infinite family of cubic 2-regular graphs, and Conder and Praeger [4] constructed two infinite families of cubic s-regular graphs for $s = 2$ or 4. Several different types of infinite families of tetravalent 1-regular graphs have been constructed in [14, 16, 21]. The first cubic 1-regular graph was constructed by Frucht [10] and later Miller [20] constructed an infinite family of cubic 1-regular graphs of order $2p$, where $p \ge 13$ is a prime congruent to 1 modulo 3. By Cheng and Oxley's classification of symmetric graphs of order $2p$ (see [2]), Miller's construction is actually the all cubic 1-regular graphs of order $2p$. By using the Marušič and Xu's results in [19], Miller's construction can be generalized to graphs of order $2n$, where $n \ge 13$ is odd such that 3 divides $\varphi(n)$, the Euler function (see [1, 18]). It may be shown that all cubic 1-regular Cayley graphs on the dihedral groups (see [18]) are exactly those graphs generalized by Miller's construction. Also, as shown in [17] or [18], one can see an importance of a study for cubic 1-regular graphs in connection with chiral (that is regular and irreflexible) maps on a surface by means of tetravalent half-transitive graphs or in connection with symmetries of hexagonal molecular graphs on the torus.

Recently, regular coverings of a graph have received considerable attention (see [6–9, 15]). An infinite family of 1-regular cyclic covering of the three-dimensional Hypercube was constructed in [7] and a classification of s-regular cyclic coverings of the complete bipartite graph $K_{3,3}$ was given in [8] for each $s \ge 1$, whose fibre-preserving automorphism subgroups act arc-transitively. However, classifications of all s-regular cyclic or elementary abelian coverings of the complete graph K_4 can be easily obtained by a method similar to this paper. Actually, such classifications were shown in [9] in a much extended family. In this paper, we classify s-regular dihedral coverings of K_4, whose fibre-preserving automorphism subgroups act arc-transitively. As a result, an infinite family of cubic 1-regular graphs is constructed, which contains those cubic 1-regular graphs constructed in [7] as a subfamily. This new family of cubic 1-regular graphs has order $8n$ such that n divides $k^2 - k + 1$ for $3 \le k \le n-2$. Following D. Marušič and T. Pisanski's classification of cubic one-regular Cayley graphs on the dihedral groups in [18], each graph in this family is not a Cayley graph on a dihedral group and so not a metacirculant graph in [1], so that it cannot belong to any family discussed in the previous paragraph.

Let k and n be non-negative integers. Let $\mathbb{Z}_n$ denote the cyclic group of order n and D_{2n} the dihedral group of order $2n$. Set

$$D_{2n} = \langle a, b \mid a^2 = b^n = 1, b^a = b^{-1} \rangle$$

and denote by $\{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ the vertex set of K_4. For $2 \le k \le n-1$, the graph $DK(k, 2n)$ is defined to have vertex set

$$V(DK(k,2n)) = \{\mathbf{w},\mathbf{x},\mathbf{y},\mathbf{z}\} \times D_{2n}$$

and edge set

$$\begin{aligned} E(DK(k,2n)) = \{&((\mathbf{w},c)(\mathbf{x},c)),\ ((\mathbf{w},c)(\mathbf{y},c)),\ ((\mathbf{w},c)(\mathbf{z},c)), \\ &((\mathbf{x},c)(\mathbf{y},ac)),\ ((\mathbf{x},c)(\mathbf{z},ab^kc)),\ ((\mathbf{y},c),(\mathbf{z},abc)) \mid c \in D_{2n}\}. \end{aligned} \tag{1.1}$$

The following is the main result of this paper.

Theorem 1.1. *Let $\widetilde{X}$ be a connected D_{2n}-covering ($n \geq 3$) of the complete graph K_4, whose fibre-preserving subgroup is arc-transitive. Then $\widetilde{X}$ is 1-regular or 2-regular. Furthermore, $\widetilde{X}$ is 1-regular if and only if $\widetilde{X}$ is isomorphic to one of $DK(k,2n)$ for $3 \leq k \leq n-2$ satisfying $n \mid (k^2-k+1)$, and $\widetilde{X}$ is 2-regular if and only if $\widetilde{X}$ is isomorphic to $DK(2,6)$.*

§ 2. Voltage Graphs and Lifting Problems

Let X be a graph and K a finite group. By a^{-1}, we mean the reverse arc to an arc a. A voltage assignment (or, K-voltage assignment) of X is a function $\phi : A(X) \to K$ with the property that $\phi(a^{-1}) = \phi(a)^{-1}$ for each arc $a \in A(X)$. The values of ϕ are called voltages, and K is the voltage group. The graph $X \times_\phi K$ derived from a voltage assignment $\phi : A(X) \to K$ has vertex set $V(X) \times K$ and edge set $E(X) \times K$, so that an edge (e,g) of $X \times_\phi K$ joins a vertex (u,g) to $(v,\phi(a)g)$ for $a = (u,v) \in A(X)$ and $g \in K$, where $e = uv$.

Clearly, the derived graph $X \times_\phi K$ is a covering of X with the first coordinate projection $p : X \times_\phi K \to X$, which is called the natural projection. By defining $(u,g')^g := (u,g'g)$ for any $g \in K$ and $(u,g') \in V(X \times_\phi K)$, K can be identified with a subgroup of $\mathrm{Aut}(X \times_\phi K)$ acting semiregularly on $V(X \times_\phi K)$. Therefore, $X \times_\phi K$ can be viewed as a K-covering. Conversely, each regular covering $\widetilde{X}$ of X with the covering transformation group K can be described as a derived graph $X \times_\phi K$. Given a spanning tree T of the graph X, a voltage assignment ϕ is said to be T-reduced if the voltages on the tree arcs are the identity. Gross and Tucker [11] showed that every regular covering $\widetilde{X}$ of a graph X can be derived from a T-reduced voltage assignment ϕ with respect to an arbitrary fixed spanning tree T of X. It is clear that if ϕ is reduced, the derived graph $X \times_\phi K$ is connected if and only if the voltages on the cotree arcs generate the voltage group K.

Let $\widetilde{X}$ be a K-covering of X with a projection p. If $\alpha \in \mathrm{Aut}(X)$ and $\widetilde{\alpha} \in \mathrm{Aut}(\widetilde{X})$ satisfy $\widetilde{\alpha}p = p\alpha$, we call $\widetilde{\alpha}$ a lift of α, and α the projection of $\widetilde{\alpha}$. Concepts such as a lift of a subgroup of $\mathrm{Aut}(X)$ and the projection of a subgroup of $\mathrm{Aut}(\widetilde{X})$ are self-explanatory. The lifts and the projections of such subgroups are of course subgroups in $\mathrm{Aut}(\widetilde{X})$ and $\mathrm{Aut}(X)$, respectively. The problem whether an automorphism α of X lifts or not can be grasped in terms of a voltage as follows. Observe that a voltage assignment on arcs extends to a voltage assignment on walks in a natural way. Given $\alpha \in \mathrm{Aut}(X)$, we define a function $\overline{\alpha}$ from the set of voltages of fundamental closed walks based at a fixed vertex $v \in V(X)$ to the voltage group K by

$$(\phi(C))^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over all fundamental closed walks at v, and $\phi(C)$ and $\phi(C^\alpha)$ are the voltages of C and C^α, respectively. Note that if K is abelian, $\overline{\alpha}$ does not depend on the choice of the base vertex, and the fundamental closed walks at v can be substituted by the fundamental cycles generated by the cotree arcs of X.

The next proposition is a special case of Theorem 4.2 in [13].

Proposition 2.1. *Let $X \times_\phi K \to X$ be a connected K-covering. Then, an automorphism α of X lifts if and only if $\overline{\alpha}$ extends to an automorphism of K.*

Two coverings $\widetilde{X}_1$ and $\widetilde{X}_2$ of X with projections p_1 and p_2 respectively, are said to be isomorphic if there exists a graph isomorphism $\widetilde{\alpha} : \widetilde{X}_1 \to \widetilde{X}_2$ such that $\widetilde{\alpha} p_2 = p_1$.

Proposition 2.2. (cf. [12]) *Two connected regular coverings $X \times_\phi K$ and $X \times_\psi K$, where ϕ and ψ are T-reduced, are isomorphic if and only if there exists an automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\phi(u, v)^\sigma = \psi(u, v)$ for any cotree arc (u, v) of X.*

§ 3. Proof of Theorem 1.1

Note that the dihedral group $D_{2n} = \{a, b \mid a^2 = b^n = 1, b^a = b^{-1}\}$ is not abelian if $n \geq 3$. As before, let $V(K_4) = \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$. Let $\widetilde{X} = K_4 \times_\phi D_{2n}$ be a covering graph of the complete graph K_4 satisfying the hypotheses in Theorem 1.1, where ϕ is a T-reduced D_{2n}-voltage assignment on a spanning tree T as illustrated by dark lines in Fig. 1, and for the cotree arcs $(\mathbf{x}, \mathbf{y})$, $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{x})$ we assign voltages z_1, z_2 and z_3 respectively. Since $K_4 \times_\phi D_{2n}$ is assumed to be connected, we have $\langle z_1, z_2, z_3 \rangle = D_{2n}$. Clearly, the

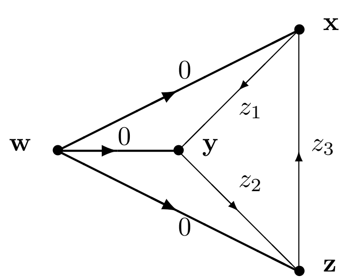

Fig. 1. The complete graph K_4 with voltage assignment ϕ

automorphism of K_4 is the symmetric group S_4 of degree 4, which is 2-regular. It is well-known that S_4 has only one 1-regular subgroup, that is, the alternating group A_4. Let

$$\alpha = (\mathbf{w}\ \mathbf{x})(\mathbf{y}\ \mathbf{z}), \quad \beta = (\mathbf{x}\ \mathbf{y}\ \mathbf{z}), \quad \gamma = (\mathbf{w}\ \mathbf{x}).$$

Then, α, β and γ are automorphisms of K_4, and we have $\alpha, \beta \in A_4$ and $\gamma \notin A_4$. By the hypotheses, the fibre-preserving subgroup, say $\widetilde{L}$, of the covering graph $K_4 \times_\phi D_{2n}$ acts arc-transitively on $K_4 \times_\phi D_{2n}$. Hence, the projection, say L, of $\widetilde{L}$ is arc-transitive on the base graph K_4. It follows that $L = A_4$ or S_4, implying that $\alpha, \beta \in L$. Thus, α and β lift.

By $i_1 i_2 \cdots i_s$, we denote a directed cycle whose vertices are $i_1, i_2, \cdots, i_s$ in a consecutive order. There are three fundamental cycles $\mathbf{wxy}$, $\mathbf{wyz}$ and $\mathbf{wzx}$ in K_4, which are generated by the three cotree arcs $(\mathbf{x}, \mathbf{y})$, $(\mathbf{y}, \mathbf{z})$ and $(\mathbf{z}, \mathbf{x})$, respectively. Each cycle maps to a cycle of same length under the actions of α, β and γ. We list all these cycles and their voltages in Table 1, in which C denotes a fundamental cycle of K_4 and $\phi(C)$ denotes the voltage on the cycle C.

C	$\phi(C)$	C^α	$\phi(C^\alpha)$	C^β	$\phi(C^\beta)$	C^γ	$\phi(C^\gamma)$
wxy	z_1	**xwz**	z_3	**wyz**	z_2	**xwy**	z_1^{-1}
wyz	z_2	**xzy**	$z_1^{-1}z_2^{-1}z_3^{-1}$	**wzx**	z_3	**xyz**	$z_3z_2z_1$
wzx	z_3	**xyw**	z_1	**wxy**	z_1	**xzw**	z_3^{-1}

Table 1. Fundamental cycles and their images with corresponding voltages on K_4

Consider the mapping $\overline{\alpha}$ from the set of voltages of the three fundamental cycles of K_4 to the voltage group K, defined by

$$\phi(C)^{\overline{\alpha}} = \phi(C^\alpha),$$

where C ranges over the three fundamental cycles. From Table 1, one can see that

$$z_1^{\overline{\alpha}} = z_3, \quad z_2^{\overline{\alpha}} = z_1^{-1}z_2^{-1}z_3^{-1}, \quad z_3^{\overline{\alpha}} = z_1.$$

(In the rest of the paper, all arithmetic operations are to be taken modulo n if at least one argument comes from the group $\mathbb{Z}_n$ and the symbol mod n is always omitted. This should cause no confusion). In a similar way, $\overline{\beta}$ and $\overline{\gamma}$ can be defined and their values can be found easily from Table 1. Since $\alpha, \beta \in L$, by Proposition 2.1, $\overline{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of K, say α^* and β^*, respectively. From Table 1, $z_1^{\beta^*} = z_2$ and $z_2^{\beta^*} = z_3$. This implies that z_1, z_2 and z_3 have the same order in the group D_{2n}.

Let $Z(D_{2n})$ denote the center of D_{2n}. An exercise shows that if n is odd $Z(D_{2n}) = 1$, and if n is even $Z(D_{2n}) \cong \mathbb{Z}_2$. Since

$$z_1^{\beta^*} = z_2, \quad z_2^{\beta^*} = z_3, \quad z_3^{\beta^*} = z_1$$

and

$$\langle z_1, z_2, z_3\rangle = D_{2n},$$

we have that z_1, z_2, z_3 are involutions, but neither in the center of D_{2n}. By Proposition 2.2, one may assume that $z_1 = a$, $z_2 = ab^i$ and $z_3 = ab^j$. It follows that

$$a^{\beta^*} = ab^i, \quad (ab^i)^{\beta^*} = ab^j, \quad (ab^j)^{\beta^*} = a.$$

Thus

$$(b^j)^{\beta^*} = (aab^j)^{\beta^*} = a^{\beta^*}(ab^j)^{\beta^*} = ab^ia = b^{-i},$$

implying that b^i and b^j have the same order. As

$$\langle a, ab^i, ab^j\rangle = D_{2n}, \quad \langle b^i, b^j\rangle = \langle b\rangle,$$

so that each of b^i and b^j generates $\langle b\rangle$. This implies that $(i, n) = 1$ and $(j, n) = 1$. Hence, $a \mapsto a$ and $b^i \mapsto b$ can deduce an automorphism of D_{2n}. Again by Proposition 2.2, one may assume that $z_1 = a$, $z_2 = ab$ and $z_3 = ab^k$ for $2 \le k < n$. Since

$$z_1^{\beta^*} = z_2, \quad z_2^{\beta^*} = z_3, \quad z_3^{\beta^*} = z_1,$$

one can deduce $k^2 - k + 1 = 0$.

If $k = 2$ then $3 = 0$, implying $n = 3$. In this case, one can easily show that $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ can be extended to automorphisms of D_{2n} and by Proposition 2.1, α, β, γ lift. By [3], there exists only one cubic symmetric graph of order 24, which is 2-regular. Thus, $\widetilde{X}$ is

2-regular. By the equation (1.1), $\widetilde{X} \cong DK(2,6)$. If $k = n-1$, then we also have $3 = 0$ and so $\widetilde{X} \cong DK(2,6)$.

As the remaining case, let $3 \le k \le n-2$ and $k^2 - k + 1 = 0$. In this case, the covering graph $K_4 \times_\phi D_{2n}$ is isomorphic to the graph $DK(k,2n)$ in the equation (1.1), where the voltage assignment ϕ is illustrated in Fig. 1 with the values $z_1 = a$, $z_2 = ab$ and $z_3 = ab^k$ in D_{2n}. To complete the proof, it suffices to show that $K_4 \times_\phi D_{2n}$ is 1-regular. Note that $k^2 - k + 1 = 0$ implies that $(k,n) = 1$ and $(k-1,n) = 1$. From Table 1, one can show that $\overline{\alpha}$ and $\overline{\beta}$ can be extended to automorphisms of D_{2n} induced by $a \mapsto ab^k$ and $b \mapsto b^{-1}$, and $a \mapsto ab$ and $b \mapsto b^{k-1}$, respectively. By Proposition 2.1, α and β lift to automorphisms of $K_4 \times_\phi D_{2n}$, which means that $K_4 \times_\phi D_{2n}$ is arc-transitive. Let $A = \mathrm{Aut}(K_4 \times_\phi D_{2n})$.

To show the 1-regularity of $K_4 \times_\phi D_{2n}$, it suffices to prove that the stabilizer of a given arc of $K_4 \times_\phi D_{2n}$ in A is trivial. For simplicity, we denote by $\mathbf{v}_c$ the vertex $(\mathbf{v}, c)$ of $K_4 \times_\phi D_{2n}$ where $\mathbf{v} \in \{\mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$ and $c \in D_{2n}$, and by $A_{\{\mathbf{w}_1,\mathbf{x}_1\}}$ the subgroup of A fixing the vertices $\mathbf{w}_1$ and $\mathbf{x}_1$. By the arc-transitivity of $\widetilde{X}$, we only need to prove $A_{\{\mathbf{w}_1,\mathbf{x}_1\}} = 1$. For $u \in V(K_4 \times_\phi D_{2n})$, let $N_i(u) = \{v \in V(K_4 \times_\phi D_{2n}) \mid d(u,v) = i\}$, where i is a non-negative integer and $d(u,v)$ is the distance between u and v. Depict the induced subgraph $\left\langle \bigcup_{i=0}^{4} N_i(\mathbf{w}_1) \right\rangle$ of $\bigcup_{i=0}^{4} N_i(\mathbf{w}_1)$ in $K_4 \times_\phi D_{2n}$, as shown in Fig. 2.

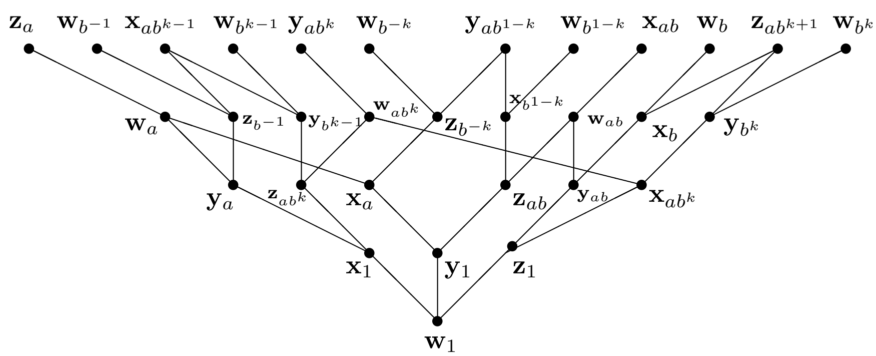

Fig. 2. Induced subgraph of $\bigcup_{i=0}^{4} N_i(\mathbf{w}_1)$ in $K_4 \times_\phi D_{2n}$

Note that $k^2 - k + 1 = 0$ implies that n is odd. Since $3 \le k \le n-2$ and $k^2 - k + 1 = 0$, we have

$$n \ge 5, \quad 2k \ne 0, \quad 2(k-1) \ne 0, \quad 2k - 1 \ne 0.$$

With these inequalities, one can show that all vertices in Fig. 2 are distinct. For the sake of convenience, we abuse the notation $i_1 i_2 \cdots i_s$ for an undirected cycle which has vertices $i_1, i_2, \cdots, i_s$, and a cycle always means an undirected one from now on. By examining Fig. 2, one can deduce that there are exactly three cycles of length 6 passing through the vertex $\mathbf{w}_1$ in $K_4 \times_\phi D_{2n}$:

$$\mathbf{w}_1\mathbf{x}_1\mathbf{y}_a\mathbf{w}_a\mathbf{x}_a\mathbf{y}_1, \quad \mathbf{w}_1\mathbf{x}_1\mathbf{z}_{ab^k}\mathbf{w}_{ab^k}\mathbf{x}_{ab^k}\mathbf{z}_1, \quad \mathbf{w}_1\mathbf{y}_1\mathbf{z}_{ab}\mathbf{w}_{ab}\mathbf{y}_{ab}\mathbf{z}_1.$$

Of these cycles, two of them pass through the edge $\mathbf{w}_1\mathbf{x}_1$. By the arc-transitivity of $K_4 \times_\phi D_{2n}$, there are exactly three cycles of length 6 passing through any vertex of $K_4 \times_\phi D_{2n}$ and there are exactly two cycles of length 6 passing through any edge of $K_4 \times_\phi D_{2n}$. For later use, we choose two cycles of length 6 passing through $\mathbf{w}_c$ or $\mathbf{x}_c$ for each $c \in D_{2n}$, respectively: say

$$C_{\mathbf{w}}(c) = \mathbf{w}_c\mathbf{y}_c\mathbf{z}_{abc}\mathbf{w}_{abc}\mathbf{y}_{abc}\mathbf{z}_c,$$
$$C_{\mathbf{x}}(c) = \mathbf{x}_c\mathbf{y}_{ac}\mathbf{z}_{b^{-1}c}\mathbf{x}_{ab^{k-1}c}\mathbf{y}_{b^{k-1}c}\mathbf{z}_{ab^kc}.$$

Now, we claim that $A_{\{\mathbf{w}_1,\mathbf{x}_1\}}$ fixes $\mathbf{y}_1$ and $\mathbf{z}_1$. Let $\omega \in A_{\{\mathbf{w}_1,\mathbf{x}_1\}}$.

First, let us assume that ω fixes $\mathbf{w}_{b^{\ell(k-2)}}$ and $\mathbf{x}_{b^{\ell(k-2)}}$. Consider the cycle $C_{\mathbf{x}}(b^{\ell(k-2)})$:

$$C_{\mathbf{x}}(b^{\ell(k-2)}) = \mathbf{x}_{b^{\ell(k-2)}}\mathbf{y}_{ab^{\ell(k-2)}}\mathbf{z}_{b^{\ell(k-2)-1}}\mathbf{x}_{ab^{(\ell+1)(k-2)+1}}\mathbf{y}_{b^{(\ell+1)(k-2)+1}}\mathbf{z}_{ab^{\ell(k-2)+k}}.$$

This is the only cycle of length 6 in $K_4 \times_\phi D_{2n}$ which passes through the vertex $\mathbf{x}_{b^{\ell(k-2)}}$, but not the edge $\mathbf{x}_{b^{\ell(k-2)}}\mathbf{w}_{b^{\ell(k-2)}}$. Therefore, ω fixes the cycle $C_{\mathbf{x}}(b^{\ell(k-2)})$ setwise and consequently, it fixes $\mathbf{x}_{ab^{(\ell+1)(k-2)+1}}$, the opposite of the vertex $\mathbf{x}_{b^{\ell(k-2)}}$ in the cycle $C_{\mathbf{x}}(b^{\ell(k-1)})$. Since the valency of $K_4 \times_\phi D_{2n}$ is 3, there is only one vertex which is adjacent to $\mathbf{x}_{ab^{(\ell+1)(k-2)+1}}$, but not on the cycle $C_{\mathbf{x}}(b^{\ell(k-2)})$. It is actually the vertex $\mathbf{w}_{ab^{(\ell+1)(k-2)+1}}$. Thus, ω also fixes $\mathbf{w}_{ab^{(\ell+1)(k-2)+1}}$. Consequently, ω fixes two vertices $\mathbf{w}_{ab^{(\ell+1)(k-2)+1}}$ and $\mathbf{x}_{ab^{(\ell+1)(k-2)+1}}$. Similarly, by considering the following cycle

$$\begin{aligned}&C_{\mathbf{w}}(ab^{(\ell+1)(k-2)+1})\\ &= \mathbf{w}_{ab^{(\ell+1)(k-2)+1}}\mathbf{y}_{ab^{(\ell+1)(k-2)+1}}\mathbf{z}_{b^{(\ell+1)(k-2)}}\mathbf{w}_{b^{(\ell+1)(k-2)}}\mathbf{y}_{b^{(\ell+1)(k-2)}}\mathbf{z}_{ab^{(\ell+1)(k-2)+1}},\end{aligned}$$

one can conclude that ω fixes $\mathbf{w}_{b^{(\ell+1)(k-2)}}$ and $\mathbf{x}_{b^{(\ell+1)(k-2)}}$.

Now, by using induction on ℓ with the hypothesis that ω fixes $\mathbf{w}_1$ and $\mathbf{x}_1$, one can obtain that ω fixes $\mathbf{w}_{b^{\ell(k-2)}}$ and $\mathbf{x}_{b^{\ell(k-2)}}$ for all non-negative integers ℓ. Clearly, this is also true for all integers ℓ because b has order n.

Let $(k-2,n) = r$. Then r divides $(k-2)^2 = k^2-4k+4$. Since $r \mid n$ and $n \mid (k^2-k+1)$, r divides

$$(k^2 - k + 1) - (k^2 - 4k + 4) = 3k - 3 = 3(k - 1).$$

Note that $n \mid (k^2 - k + 1)$ implies $(n, k - 1) = 1$. As $r \mid n$, $(r, k - 1) = 1$. Thus, $r \mid 3(k - 1)$ implies $r \mid 3$, namely $(k - 2, n) = 1$ or 3.

Case I. Let $(k-2,n) = 1$. Then, there exist integers s and t such that $s(k-2)+tn = 1$. Thus, ω fixes $\mathbf{x}_{b^{s(k-2)}} = \mathbf{x}_{b^{1-tn}} = \mathbf{x}_b$. On the other hand, since ω fixes $\mathbf{w}_1$ and $\mathbf{x}_1$, it also fixes $\{\mathbf{y}_1, \mathbf{z}_1\}$ setwise. As $d(\mathbf{x}_b, \mathbf{z}_1) = 2$ and $d(\mathbf{x}_b, \mathbf{y}_1) \neq 2$ (see Fig. 2), it follows that ω fixes $\mathbf{y}_1$ and $\mathbf{z}_1$ pointwise.

Case II. Let $(k-2,n) = 3$. Then, there exist integers s and t such that $s(k-2)+tn = 3$. It follows that ω fixes $\mathbf{x}_{b^{(s+1)(k-2)}} = \mathbf{x}_{b^{k+1}}$. From Fig. 2, one can see that

$$N_4(\mathbf{w}_1) = \{\mathbf{z}_a, \mathbf{w}_{b^{-1}}, \mathbf{x}_{ab^{k-1}}, \mathbf{w}_{b^{k-1}}, \mathbf{y}_{ab^k}, \mathbf{w}_{b^{-k}}, \mathbf{y}_{ab^{1-k}}, \mathbf{w}_{b^{1-k}}, \mathbf{x}_{ab}, \mathbf{w}_b, \mathbf{z}_{ab^{k+1}}, \mathbf{w}_{b^k}\},$$
$$N_3(\mathbf{x}_{b^{k+1}}) = \{\mathbf{x}_{ab^{k+1}}, \mathbf{z}_{ab^{k+2}}, \mathbf{y}_{ab^{k+2}}, \mathbf{x}_{ab^{2k+1}}, \mathbf{z}_{ab^{k+1}}, \mathbf{w}_{b^k}, \mathbf{x}_{ab^{2k}}, \mathbf{w}_{b^{2k}}, \mathbf{y}_{ab^{2k+1}}\}.$$

Since $3 \leq k \leq n-2$, we have $n \geq 5$ and since $(k-2,n) = 3$ and $n \mid (k^2-k+1)$, it is easy to show that $k+1 \neq 0,\ k+2 \neq 0,\ 2k \neq 0,\ 2k+1 \neq 0,\ 2k-1 \neq 0,\ 3k \neq 0$ and $3k-1 \neq 0$. With these inequalities, one may obtain that $N_3(\mathbf{x}_{b^{k+1}}) \cap N_4(\mathbf{w}_1) = \{\mathbf{w}_{b^k}, \mathbf{z}_{ab^{k+1}}\}$. Since ω fixes $\mathbf{x}_{b^{k+1}}$ and $\mathbf{w}_1$, it fixes $\{\mathbf{w}_{b^k}, \mathbf{z}_{ab^{k+1}}\}$ setwise. However, $\mathbf{z}_{ab^{k+1}}$ is on a cycle of length 6 passing through $\mathbf{z}_1$, but neither of $\{\mathbf{w}_{b^k}, \mathbf{z}_{ab^{k+1}}\}$ is on a cycle of length 6 passing through $\mathbf{y}_1$. Hence, the condition that ω fixes $\{\mathbf{y}_1, \mathbf{z}_1\}$ setwise implies that it fixes $\mathbf{y}_1$ and $\mathbf{z}_1$ pointwise.

So far, we have proved that $A_{\{\mathbf{w}_1,\mathbf{x}_1\}}$ fixes $\mathbf{y}_1$ and $\mathbf{z}_1$. Of the cycles of length 6 passing through $\mathbf{w}_1$, there is only one of them passing through any two vertices of $\mathbf{x}_1$, $\mathbf{y}_1$ and $\mathbf{z}_1$. Hence, ω fixes the three cycles of length 6 passing through $\mathbf{w}_1$ pointwise, which means that ω fixes all vertices in $N_2(\mathbf{w}_1)$. By the arc-transitivity and the connectivity of $K_4 \times_\phi D_{2n}$, we have $A_{\{\mathbf{w}_1,\mathbf{x}_1\}} = 1$, as required.

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