Journal of Mathematical Research & Exposition
Vol.24, No.2, 191–197, May, 2004

Asymptotic Flatness of Stochastic Flow on Manifolds *

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Abstract: The aim of this article is to discuss a volume nullification property of the diffusion process determined by a stochastic differential equation on a manifold. Let $X_t(x)$ be a diffusion process describing a flow of diffeomorphisms $x \to X_t(x)$ in a manifold $\mathcal{M}$, and K be a compact surface in $\mathcal{M}$ with positive finite Hausdorff measure. We present conditions under which the area of $X_t(K)$ goes to zero almost surely and in moments as $t \to \infty$, in particular, the flow $X_t(\cdot)$ asymptotic nullifies the arc-lenth of oriented rectifiable arcs $r : [0,1] \to \mathcal{M}$.

Key words: diffusion process; stochastic flow; Hausdorff measure; manifold.

Classification: AMS(2000) 60D05, 58E03/CLC number: O211.6

Document code: A **Article ID:** 1000-341X(2004)02-0191-07

1. Introduction

Asymptotic flatness of stochastic flows is an important property. It has been discussed by many authors, see [2],[6] and [9], etc. Especially, in [1], a volume nullification property of the n-dimensional singular diffusion process is discussed. Under the manifold setup, in this paper, we discuss the same kind property of the process $X_t(x,w)$ that arises as the solution of a stochastic differential equations on a manifold.

Consider the stochastic differential equation (SDE) on an m-dimensional manifold $\mathcal{M}$:

$$\begin{cases} dX_w(t) = \sum_{\alpha=1}^d A_\alpha(X_w(t)) \circ dW^\alpha(t) + A_0(X_w(t))\mathrm{d}t, \\ X_w(0) = x, \end{cases} \tag{1}$$

where $A_0, A_1, \cdots, A_d \in \mathcal{X}(\mathcal{M})$ (the space of tangent vector fields), $\circ dW^\alpha$ denotes the stochastic differential in Stratonovich's sense, and $\{W^\alpha, \alpha = 1, 2, \cdots, d\}$ is d-dimensional standard Brownian motion.

*Received date: 2002-06-03
Foundation item: Supported by the key Project of Chinese Minstry of Education and Supported by Natural Science Foundation of Beijing (1022004)
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Now we choose a local coordinate set $U \ni x$, and assume that U is relatively compact in $\mathcal{M}$. We then have

$$A_\alpha = \sum_{i=1}^{m} \sigma_\alpha^i \frac{\partial}{\partial x}, \quad \alpha = 0, 1, \cdots, d$$

extending σ_α^i to a smooth function $\tilde{\sigma}_\alpha^i$ with compact support in $\mathbf{R}^m$.

Consider the SDE in $\mathbf{R}^m$:

$$\begin{cases} dX^i(t) = \sum_{\alpha=1}^{d} \tilde{\sigma}_\alpha(X_t) \circ dW^\alpha(t) + \tilde{\sigma}_0^i(X_w(t))\mathrm{d}t, \\ X_0 = x, \end{cases} \tag{2}$$

which is equivalent to the following one

$$\begin{aligned} dX^i(t) &= \sum_{\alpha=1}^{d} \tilde{\sigma}_\alpha(X_t) \cdot dW^\alpha(t) + (\tilde{\sigma}_0^i + \frac{1}{2}\sum_{\alpha=1}^{d}(\frac{\partial}{\partial x^k} \cdot \tilde{\sigma}_\alpha^i) \cdot \tilde{\sigma}_\alpha^k)(X_t)\mathrm{d}t, \\ X_0 &= x, \end{aligned} \tag{3}$$

where $\cdot dW^\alpha(t)$ denotes the differential in Itó's sense.

From the assumption we have made above, we can see that the coefficient of (3) satisfies the all conditions of existence and uniqueness theorem[10]. Hence the equation (3) has a unique solution $X_t(x, w)$. Let

$$\tau_U(w) = \inf\{t > 0, X_t(x, w) \notin U\}.$$

Then we consider the following two cases

1. $\tau_U(w) = \infty$. This means that the path of $X_t(x, w)$ would stay in U forever.

2. $\tau_U(w) < \infty$. This means that the path would go out of U at some finite time t.

Asymptotic flatness of stochastic flows has been well discussed. Especially, in [1], a volume nullification property of the n-dimensional singular diffusion process is discussed. In this paper, we discuss the same kind property of the process $X_t(x, w)$ that arises as the solution of the SDE (1).

First, we know that the flow $t \to X_t(x, w)$ is a flow of diffeomorphisms $x \to X_t^x$ in $\mathcal{M}$ (see [10]). Let K be a compact surface in $\mathcal{M}$ with finite Hausdorff measure (here we should consider two cases: one is looking K as a subset of a higher dimensional Euclidean spaceR^n when $\mathcal{M}$ is a general manifold,the other is considering K's Hausdorff measure as a $\mathcal{L}^n$-measurable subset of $\mathcal{M}$ when $\mathcal{M}$ is a Riemannian manifold with a Riemannian metric g, where $\mathcal{L}^n$ is the Lebesgue mesure of $\mathbf{R}^n$) and $\varphi : B \subset \mathbf{R}^r \to K, r \leq n$ be its Lipschitzian parametrization (here we suppose $\mathcal{M}$ is metrizable). We can take $B := [0,1]^r$. We present conditions so that the area of $(X_t \circ \varphi)(B)$ goes to zero almost surely and in L^p as $t \to \infty$. In particular, for a rectifiable arc $c : [0,1] \to \mathcal{M}$. We get the asymptotic nullification of the arc length of $X_t(c)$. From this we recover the asymptotic flatness of $X_t(x)$.

Under the coordinate system of U, the Jacobian matrix of the diffeomorphism $x \to X_t(x)$: $Y_t(x) := (\partial_j X_t(x))$, by Itó's formula:

$$Y_t(x) = I + \int_0^t \Sigma_\alpha \sigma'_\alpha(X_s(x))Y_s(x)dW_s^\alpha + \int_0^t b'(X_s(x))y_s(x)\mathrm{d}s, \tag{4}$$

where

$$b^i(X_s(x)) = \frac{1}{2}\Sigma_\alpha(\frac{\partial}{\partial x^k}\sigma_\alpha^i)\sigma_\alpha^k(X_s(x)). \tag{5}$$

Now we will introduce some notations and formulas that would be used in the following.

Let $M_{(m\times m)}$ denote $m \times m$-dimensional matrix, and $\langle\cdot,\cdot\rangle_{(m\times m)}$ be the trace norm of $m \times m$ dimensional matrix. Let $(\cdot,\cdot)_g$ be the norm of a Riemannian manifold $\mathcal{M}$ induced by the Riemannian metric g. The notions of area are from differential geometry and geometric measure theory.

First of all, let us consider the simplest case: $\mathbf{R}^m$ set up. Recall the definition of r-dimensional Hausdorff measure. For an arbitrary subset $B \subset \mathbf{R}^m$, let $\delta(B)$ denote the diameter of B. By $\beta(r)$ we shall denote the Lebesgue measure of closed unit ball in $\mathbf{R}^r$, i.e., $\beta(r) = (\Gamma(\frac{1}{2}))^r/(\Gamma(\frac{r}{2}+1))$. For any subset $A \subset \mathbf{R}^m$, one defines the r-dimensional Haudorff measure $\mathcal{H}^r(A)$ as follows: For a small $\eta > 0$ cover A by countably many subsets S_i with $\delta(S_i) < \eta$, and define

$$\mathcal{H}^r(A) := \lim_{\eta\to 0} \inf_{A\subset\cup S_i, \delta(S_i)\leq\eta} \sum_i 2^{-r}\beta(r)(\delta(S_i))^r.$$

One can show that, for any positive integer m, $\mathcal{H}^m = \mathcal{L}^m$, where $\mathcal{L}^m$ is the Lebesgue measure on $\mathbf{R}^m$. To deal with (non smooth) surfaces in $\mathbf{R}^m$, one works with Lipschitz functions $f : \mathbf{R}^r \to \mathbf{R}^m$; If $f : \mathbf{R}^r \to \mathbf{R}^m$ is differentiable at $a \in \mathbf{R}^r$, one defines the k-dimensional Jacobian $J_k f(a)$ of f at a as the maximum k-dimensional volume of the image under $Df(a)(\equiv f'(a))$ of a unit k-dimensional cube. Note that if $k = r = m$, then $J_k f(a) = |\det Df(a)|$. Finally, we recall the area formula.

Area Formula: For an $\mathcal{L}^n$-measurable set A and Lipschitz functions $f : \mathbf{R}^r \to \mathbf{R}^m$ with $r \leq m$

$$\int_{\mathbf{R}^r} I_A(x)J_r f(x)\mathcal{L}^r(x) = \int_{\mathbf{R}^m} \sum_{x\in f^{-1}(y)} I_A(x)\mathrm{d}\mathcal{H}^r(y).$$

Secondary, we state the result when it comes to a Riemannian manifold without proof. In this article, we shall consider the case when the manifold has only one coordinate system. Under the Riemannian metric g we can give a distance ρ induced by g, i.e.,

$$\rho(x,y) = \inf_{c\in C^l} \int_K (c'(t), c'(t))_g \mathrm{d}\mathcal{L}_t,$$

where C^l denotes the set of Lipschitzian function c mapping some compact interval K into $\mathcal{M}$ with $c(\inf K) = x, c(\sup K) = y$. Then $\mathcal{M}$ is a metric space with ρ.

Let $\mathcal{H}^r(B)$ denote the r-dimensional Hausdorff measure corresponding to ρ, where B is a r-dimensional subset of $\mathcal{M}$ and we have an Area on manifold which is similar to that of in $\mathbf{R}^m$.

For a $\mathcal{L}^m$-measurable set A and Lipschitz functions $f : \mathbf{R}^r \to \mathcal{M}$, with $r \leq m$, we have

$$\int_{\mathbf{R}^r} I_A(x) J_r f(x) \mathcal{L}^r(x) = \int_{\mathcal{M}} \sum_{x \in f^{-1}(y)} I_A(x) \mathrm{d}\mathcal{H}^r_\rho(y).$$

Now, there are two lemmas as follows.

Lemma 1 If $\tau_U(\omega) = \infty$, make the assumption (A): There exists a $\lambda > 0$ and a positive definite matrix $C \in M_{(m\times m)}$ such that, for all $Z \in M_{(m\times m)}$

$$\begin{aligned}&\langle CZ, \{2b'(x) + \sum_{\alpha=1}^{d} C^{-1}\sigma'_\alpha(x) C \sigma'_\alpha(x)\} Z\rangle_{(m\times m)} - 2\sum_\alpha \frac{(\langle CZ, \sigma'_\alpha(x) Z\rangle_{(m\times m)})^2}{\langle CZ, Z\rangle_{(m\times m)}} \\ &\leq -\lambda \langle Z, Z\rangle_{(m\times m)}.\end{aligned}$$

Let $\|\cdot\|_{m\times m}$ denote the corresponding norm induced by $\langle\cdot,\cdot\rangle_{(m\times m)}$. Then there exists a $\delta \in [0,1)$ such that $\|Y_t(x)\|^{2(1-\delta)} \to 0$ exponentially fast in L^1 and almost surely at $t \to \infty$.

Proof Notice $\tau_U = \infty$ means that the path of solutions of (1) always stay in U. Thus (U, φ, x^i) is a global coordinate system of U. We can restrict the equation (1) on U, which has the same solution in the form as that of equation (2) when it is restricted on $\varphi(U)$. Hence the result can be deduced from case in $\mathbf{R}^m$. From Theorem 3.2 in [1], the proof is complete.

Lemma 2 *Suppose $\tau_U(\omega) = \infty$. Modify the Assumption (A) as following:*

Assumption (B): for some integer $l \geq 1$ there exists a $\lambda > 0$ and a positive definite matrix $C \in M_{(m\times m)}$ such that for all $m \times m$ matrix Z

$$\begin{aligned}&\langle CZ, \{2b'(x) + \sum_\alpha C^{-1}\sigma'^T_\alpha(x) C \sigma'_\alpha(x)\} Z\rangle_{(m\times m)} + 2(\frac{l}{2} - 1)\sum_\alpha \frac{(\langle CZ, \sigma'_\alpha(x) Z\rangle_{(m\times m)})^2}{\langle CZ, Z\rangle_{(m\times m)}} \\ &\leq -\lambda \langle Z, Z\rangle_{(m\times m)}.\end{aligned}$$

Then $\|Y_t(x)\|^l_{(m\times m)} \to 0$ exponentially fast in L^1 and almost surely as $t \to \infty$.

Proof Similar to the discussion in Lemma 1, it is not difficult to prove the Lemma by using Theorem 3.4 in [1].

2. Main results

Theorem 1 *Let K be a compact r-dimensional surface in U such that $0 < \mathcal{H}^r(\varphi(K)) < \infty$ for some $r \leq l \leq m$, where $\mathcal{H}^r$ is the r-dimensional Hausdorff measure corresponding to the ordinary metric of $\mathbf{R}^m$ and l is mentioned in Lemma 2. Let ψ be a Lipschitzian map from $\mathbf{R}^r$ to $\mathbf{R}^m$, $B = \psi^{-1}(\varphi(K))$. Suppose $\tau_U(K,\omega) = \inf_{x\in K}\{t : X_t(x) \notin U\} = \infty$. Then under the assumption of Lemma 1, Area$[(X_t(\circ\psi)(B)] \to 0$ in L^1 and almost surely exponentially fast as $t \to \infty$.*

Proof For $\tau_U(\omega) = \infty$, we know the equation's solution only depend on σ and b not $\tilde{\sigma}$

and $\tilde{b}$ for the path would never go out of U. Then we can use Theorem (3.2) in Kannan's paper [1] to obtain the result.

Remark Using φ we can get a metric of U induced by the inner product of $\mathbf{R}^m$. Then U can be metric. Under this metric we know that φ is a isometric mapping. Hence all the result of $\varphi(K)$ can be transfered to $K \subset U$, i.e., K has the same Hausdorff measure as $\varphi(K)$, when we treat U as a subset of $\mathbf{R}^m$.

From the discussion above, we can see the importance of coordinate system. We can write $Y_t(x)$ only under a definite coordinate, and the form of $Y_t(x)$ will be changed if the coordinate is changed. So, when $\tau_U < \infty$, we could not discuss the problem in a general manifold. Now we will consider a Riemannian manifold $(\mathcal{M}, g)$, which it has a globally coordinate. We will consider the nullification property of the equation (1) on $(\mathcal{M}, g)$, corresponding to the Hausdorff measure induced by g.

Cantan-Hardamard Theorem[5] *Suppose $(\mathcal{M}, g)$ is a Riemannian manifold and*

1. *$\mathcal{M}$ is complete.*
2. *$\mathcal{M}$ is simply connected.*
3. *The section curvature of $\mathcal{M}$ is negative, i.e. $k_{\mathcal{M}} \leq 0$.*

Then $\mathcal{M}$ is diffeomorphism with $\mathbf{R}^m$, the diffeomorphism mapping is

$$\exp_p : \mathbf{R}^m = T_p\mathcal{M} \to \mathcal{M}$$

with p selected on $\mathcal{M}$ arbitrary.

Suppose $(\mathcal{M}, g)$ satisfies those conditions in Cantan-Hardamard Theorem. Choose $p \in \mathcal{M}$ arbitrary. Then under the assumption in Lemma 1, we know $\mathrm{Area}[X_t \circ \varphi^{-1} \circ \psi(B)] \to 0$ exponentially fast in L^1 and almost surely as $t \to \infty$, and $(T_p\mathcal{M}, \exp_p^{-1}, x^i)$ is the normal coordinate system of $\mathcal{M}$, where $\{x^i\}$ is the classical coordinate system of $\mathbf{R}^m$ under this coordinate, we can write the equation (1) as

$$\begin{cases} dX_t^i = \sum_{\alpha=1}^d \sigma_\alpha(X_t) \circ dW_t^\alpha + \sigma_0^i(X_t)dt, \\ X_0 = x_0, \end{cases} \tag{6}$$

and derive that $Y_t(x_0) = (\frac{\partial}{\partial x^j}(X_t^i(x_0))$ as (4). Note that $\mathcal{M}$ is a metric space with ρ, where ρ is the distance induced by g.

Lemma 3 *Assume that the manifold $(\mathcal{M}, g)$ satisfies all assumptions in Lemma 1 and the conditions in Cartan-Hardmard Theorem. Then*

$$||Y_t(x_0)||^{2(1-\delta)} \to 0, \quad as \quad t \to \infty,$$

where $\delta \in [0, 1)$.

Proof Since the manifold $(\mathcal{M}, g)$ has a globally coordinate system, we know that there is a unique formula (6) of SDE(1). So we can see that $\mathcal{M}$ is the same as U in Lemma 1, and hence obtain the lemma.

Lemma 4 *Suppose $(\mathcal{M}, g)$ is the same as in Lemma 3, and assume that the assumption (B) in Lemma 2 is satisfied. Then $||Y_t(x_0)||^l \to 0$ as $t \to \infty$.*

The proof is as the same as that in Lemma 3.

Theorem 2 *Let K be a compact r-dimensional surface in $\mathcal{M}$ such that $0 < \mathcal{H}_g^r(K) < \infty$ for some $r \le l \le m$, where $\mathcal{H}_g^r$ is the r-dimensional Hausdorff measure induced by the Riemannian metric g and m is the same as in Lemma 4. Let ψ be a Lipschitzian map from $\mathbf{R}^r$ to $\mathcal{M}, B = \psi^{-1}(K)$. Assume that all conditions in Lemma 3 are satisfied. Then Area$[(X_t \circ \psi)(B)] \to 0$ in L^1 and almost surely exponentially fast as $t \to \infty$.*

Proof Let $\mathcal{L}^r$ be the Lebesgue measure on $\mathbf{R}^r$. Then we have

$$\begin{aligned}\mathrm{Area}(X_t \circ \psi(B)) &= \int_K \mathrm{card}(\psi^{-1} \circ X_t(y))\mathcal{H}_g^r(\mathrm{d}y)\\ &= \int_B J_r(X_t(\psi(u))\mathcal{L}^r(\mathrm{d}u)\\ &\le C\int_B K_r\|Y_t(\psi(u))\|^r\mathcal{L}^r(\mathrm{d}u).\end{aligned}$$

where K_r is a positive constant independent of u or t. From Lemma 3, the right hand side of the last inequality goes to zero in L^1 exponentially fast as $t \to \infty$. Similar to [1], we can prove the almost surely convergence exponentially fast.

Corollary *Let c be an oriented rectifiable curve in $\mathcal{M}$. Under the assumption of Theorem 2, we have Arc-length $(X_t(c)) \to$ in L^1 and almost surely, exponentially fast as $n \to \infty$.*

The corollary is the case with $r = 1$ in Theorem 2.

A general compact manifold $\mathcal{M}$ has no globally coordinate system. But every manifold can be embedded into a higher dimensional Euclidean space. Therefore we can use the Whitney's theorem to get a coordinate system covering $\mathcal{M}$.

Consider an m-dimensional manifold $\mathcal{M}$ and an SDE on $\mathcal{M}$ with the form (1). Then we can imbed $\mathcal{M}$ into $\mathbf{R}^n$ where $n \ge m$. So there exists the vector field $\tilde{A}_0, \tilde{A}_1, \cdots, \tilde{A}_d$ in $\mathbf{R}^n$ with compact support such that $\tilde{A}_i(x) = A_i(x), \forall x \in \mathcal{M}$. Consider the SDE in $\mathbf{R}^r$

$$\begin{cases} d\tilde{X}_t = \sum_{\alpha=1}^d \tilde{A}_\alpha(\tilde{X}_t) \circ dW^\alpha(t) + \tilde{A}_0(\tilde{X}_t)\mathrm{d}t, \\ \tilde{X}_0 = x_0, \end{cases} \tag{7}$$

where $x_0 \in \mathcal{M}$. We know that (7) has a solution as the same as that for equation (1). Since the uniqueness of the solution of SDE in $\mathbf{R}^n$, we can write $\tilde{A}_\alpha|_{\mathcal{M}} = A_\alpha, \alpha = 0, 1, \cdots, d$ as

$$A_\alpha = \sum_{i=1}^n \sigma_\alpha^i \frac{\partial}{\partial x_i}.$$

We have the following result.

Theorem 3 *Let K be a compact N-dimensional surface in $\mathcal{M}$ such that $0 < \mathcal{H}^r(K) < \infty$ for some $r \le l \le n$, where $\mathcal{H}^r$ is the r-dimensional Hausdorff measure of K, and K is a subset of $\mathbf{R}^n$. Then Area$[(X_t \circ \varphi)(B)] \to 0$ in L^1 and almost surely exponentially fast as $t \to \infty$.*

Acknowlgement The authors would like to express their thanks to Professor Kannan for his helpful suggestion.

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流形上随机流的渐近平坦性

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摘 要: 本文的目的是讨论流形上由随机微分方程确定的扩散过程的体积零化性质。令 $X_t(x)$ 是描述流形 $\mathcal{M}$ 上的微分同胚流 $x \to X_t(x)$ 的扩散过程， K 是 $\mathcal{M}$ 中具有正有限 Hausdorff 测度的紧致曲面. 我们给出 $X_t(K)$ 的面积在 $t \to \infty$ 时几乎必然趋于零的条件，特别地，随机流 $X_t(\cdot)$ 的渐近零化定向可求长弧 $r:[0,1] \to \mathcal{M}$ 的弧长.

关键词: 扩散过程；随机流； Hausdorff 测度；流形.