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ON A CLASS OF WEAK BERWALD (α,β)-METRICS

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Abstract: We study an important class of weak Berwald (α,β)-metrics in the form $F = \alpha+\varepsilon\beta+\beta\ \arctan(\frac{\beta}{\alpha})$ (ε is a constant) on a manifold. By using a formula of the S-curvature, we obtain sufficient and necessary conditions for such metrics to be weak Berwald metrics. We also prove that F is a weak Berwald metric with scalar flag curvature if and only if it is a Berwald metric and its flag curvature vanishes.

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1 Introduction

In Finsler geometry, there are several important classes of Finsler metrics. The Berwald metrics were first investigated by L. Berwald. By definition, a Finsler metric F is a Berwald metric if the spray coefficients $G^i = G^i(x, y)$ are quadratic in $y \in T_xM$ at every point x, i.e.,

$$G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k.$$

Riemannian metrics are special Berwald metrics. In fact, Berwald metrics are "almost Riemannian" in the sense that every Berwald metric is affinely equivalent to a Riemannian metric, i.e., the geodesics of any Berwald metric are the geodesics of some Riemannian metric. Weak Berwald spaces were first investigated by Bácsó and Yoshikawa in 2002 [2]. The class of weak Berwald metrics is more generalized than Berwald metrics in [6]. Hence it becomes an important and natural problem to study weak Berwald (α,β)-metrics. Cui obtained the necessary and sufficient conditions for two important kinds of (α,β)-metrics in the forms of $F = \alpha+\varepsilon\beta+k\frac{\beta^2}{\alpha}$ and $F = \frac{\alpha^2}{\alpha-\beta}$ to be weak Berwald metrics in [7]. Xiang and Cheng characterized a special class of weak Berwald (α,β)-metrics in the form of $F = (\alpha+\beta)^{m+1}/\alpha^m$ in [10]. Further, Cheng and Lu studied two kinds of weak Berwald metrics of scalar flag curvature in [4].

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The purpose of this paper is to study a special class of weak Berwald (α,β)-metrics in the form of $F=\alpha+\varepsilon\beta+\beta\arctan(\frac{\beta}{\alpha})$. We have the following:

Theorem 1.1 Let $F=\alpha+\varepsilon\beta+\beta\ \arctan(\frac{\beta}{\alpha})$ be an arctangent Finsler metric on an n-dimensional manifold $M(n\geq 3)$, where ε is a constant. Then the following are equivalent:

(a) F has isotropic S-curvature, i.e., $\mathbf{S}=(n+1)cF$;

(b) F has isotropic mean Berwald curvature, i.e., $\mathbf{E}=\frac{n+1}{2}cF^{-1}h$;

(c) β is a Killing 1-form of constant length with respect to α, i.e., $r_{00}=s_0=0$;

(d) F has vanished S-curvature, i.e., $\mathbf{S}=0$;

(e) F is a weak Berwald metric, i.e., $\mathbf{E}=0$,

where $c=c(x)$ is a scalar function on M.

By [2], an arctangent Finsler metric $F=\alpha+\varepsilon\beta+\beta\ \arctan(\frac{\beta}{\alpha})$ is of scalar flag curvature with vanishing S-curvature if and only if its flag curvature $\mathbf{K}=0$ and it is a Berwald metric. In this case, F is a locally Minkowski metric. Thus F is a weak Berwald metric with scalar flag curvature, its local structure can be completely determined.

Corollary 1.2 Let $F=\alpha+\varepsilon\beta+\beta\ \arctan(\frac{\beta}{\alpha})$ be an arctangent Finsler metric on an n-dimensional manifold M $(n\geq 3)$, where ε is a constant. Then F is a weak Berwald metric with scalar flag curvature $\mathbf{K}=\mathbf{K}(x,y)$ if and only if it is a Berwald metric and $\mathbf{K}=0$. In this case, F must be locally Minkowskian.

2 Preliminaries

In Finsler geometry, (α,β)-metrics form a very important and rich class of Finsler metrics. An (α,β)-metric is expressed as the following form

$$F=\alpha\phi(s),\ s=\frac{\beta}{\alpha},$$

where α is a Riemannian metric and β is a 1-form. $\phi(s)$ is a positive C^∞ function on an open interval $(-b_0,b_0)$ and satisfying

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\ |s|\leq b<b_0,$$

where $b:=\|\beta\|_\alpha$. It is known that $F=\alpha\phi(s)$ is a Finsler metric if and only if $\|\beta\|_\alpha<b_0$ for any $x\in M$ in [6]. In this paper, we consider a special (α,β)-metric in the following form:

$$F=\alpha+\varepsilon\beta+\beta\arctan(\frac{\beta}{\alpha}), \tag{2.1}$$

where ε is an arbitrary constant. We call this metric an arctangent Finsler metric. Let $b_0>0$ be the largest number such that

$$\frac{1-s^2+2b^2}{(1+s^2)^2}>0,\quad |s|\leq b<b_0, \tag{2.2}$$

so that $F=\alpha+\varepsilon\beta+\beta\ \arctan(\frac{\beta}{\alpha})$ is a Finsler metric if and only if β satisfies that $\|\beta\|_\alpha<b_0$ for any $x\in M$. Let $\nabla\beta=b_{i|j}dx^i\otimes dx^j$ denote covariant derivative of β with respect to α.

Denote

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}),\ r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}),\ s_{l0} := s_{li}y^i,\ s_0 := b^l s_{l0},$$
$$r_{00} := r_{ij}y^iy^j,\ r_i := r_{ij}b^j,\ r_0 := r_jy^j.$$

Let $G^i(x,y)$ and $G^i_\alpha(x,y)$ denote the spray coefficients of F and α, respectively. We have the following formula for the spray coefficients $G^i(x,y)$ of F,

$$G^i = G^i_\alpha + \alpha Q s^i{}_0 + \Theta\{-2\alpha Q s_0 + r_{00}\}\frac{y^i}{\alpha} + \Psi\{-2\alpha Q s_0 + r_{00}\}b^i, \tag{2.3}$$

where

$$\begin{aligned} Q &= \varepsilon + \varepsilon s^2 + \arctan(s) + s^2\arctan(s) + s,\\ \Theta &= \frac{\varepsilon - s - \varepsilon s^2 - \arctan(s)\ s^2 + \arctan(s)}{2(1+2b^2-s^2)(1+\varepsilon s + s\arctan(s))},\\ \Psi &= \frac{1}{1+2b^2-s^2}. \end{aligned} \tag{2.4}$$

As is well known, the Berwald tensor of a Finsler metric F with the spray coefficients G^i is defined by $\mathbf{B_y} := B^i_{jkl}(x,y)dx^j \otimes dx^k \otimes dx^l \otimes \partial_i$, where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}. \tag{2.5}$$

Furthermore, the mean Berwald tensor $\mathbf{E_y} := E_{ij}(x,y)dx^i \otimes dx^j$ is defined by

$$E_{ij} := \frac{1}{2}B^m_{mij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}(\frac{\partial G^m}{\partial y^m}). \tag{2.6}$$

A Finsler metric is called a Berwald metric if the Berwald curvature $\mathbf{B} = 0$. A Finsler metric is called a weak Berwald metric if the mean Berwald curvature $\mathbf{E} = 0$.

The S-curvature $S = S(x,y)$ is one of the most important non-Riemannian quantities. For a Finsler metric $F = F(x,y)$ on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F dx^1 \wedge \cdots \wedge dx^n$ is given by

$$\sigma_F(x) := \frac{\mathrm{Vol}(B^n(1))}{Vol\{(y^i) \in R^n \mid F(x,y) < 1\}}.$$

Here Vol denotes the Euclidean volume in R^n. The well-known S-curvature is given by

$$S(x,y) = \frac{\partial G^m}{\partial y^m} - y^m \frac{\partial(ln\sigma_F)}{\partial x^m}.$$

Cheng and Shen obtained a formula for the S-curvature of an (α,β)-metric on an n-dimensional manifold M as follows

Lemma 2.1 [5] The S-curvature of an (α,β)-metric is given by

$$\mathbf{S} = \lambda(r_0 + s_0) + 2(\Psi + QC)s_0 + 2\Psi r_0 - \alpha^{-1}Cr_{00}, \tag{2.7}$$

where $\lambda := -\frac{f'(b)}{bf(b)}$ is a scalar function on M and $C := -(b^2 - s^2)\Psi' - (n+1)\Theta$.

3 Proof of Theorem 1.1

The proof contains the following steps:

Step 1 (a) $\Rightarrow$ (b) In fact, (a) $\Rightarrow$ (b) is obvious true.

Step 2 (b) $\Rightarrow$ (a) Assume that (b) holds, which is equivalent to

$$\mathbf{S} = (n+1)\{cF + \eta\}, \tag{3.1}$$

where η is a 1-form on M. So (a) is equivalent to (b) if and only if $\eta = 0$. Plugging (2.4) and (2.7) into (3.1), we obtain

$$\begin{aligned}
&(J_6\alpha^6 + J_5\alpha^5 + J_4\alpha^4 + J_3\alpha^3 + J_2\alpha^2 + J_1\alpha + J_0)\arctan^2(\frac{\beta}{\alpha})\\
&+(K_6\alpha^6 + K_5\alpha^5 + K_4\alpha^4 + K_3\alpha^3 + K_2\alpha^2 + K_1\alpha + K_0)\arctan(\frac{\beta}{\alpha})\\
&+M_7\alpha^7 + M_6\alpha^6 + M_5\alpha^5 + M_4\alpha^4 + M_3\alpha^3\\
&+M_2\alpha^2 + M_1\alpha + M_0 = 0,
\end{aligned} \tag{3.2}$$

where

$$\begin{aligned}
J_6 &= 2\nu s_0(1+2b^2),\ J_5 = 2\nu c\beta^2(1+2b^2)^2,\\
J_4 &= 2s_0\beta^2(4b^2-\nu),\ J_3 = -4\nu c\beta^4(1+2b^2),\\
J_2 &= -2s_0\beta^4(-2b^2+5+2nb^2+n),\ J_1 = 2c\nu\beta^6,\\
J_0 &= 2s_0\beta^6(n-3),\\
K_6 &= 4(1+2b^2)(2\nu cb^2\beta + \nu c\beta + s_0\varepsilon + \varepsilon ns_0),\\
K_5 &= -8r_0b^2\beta + 8\nu\eta b^2\beta - \nu r_{00} + 8\lambda r_0b^2\beta + 4\nu c\varepsilon\beta^2 + 2\lambda s_0\beta\\
&\quad +8\nu\eta b^2\beta + 8\lambda r_0b^4\beta + 8\lambda s_0b^2\beta + 2\nu\eta\beta - 2\nu b^2r_{00} - 4r_0\beta\\
&\quad -4s_0\beta + 8\lambda b^4s_0\beta + 16\nu c\varepsilon b^2\beta^2 + 2\lambda r_0\beta + 16\nu c\varepsilon b^4\beta^2,\\
K_4 &= -4\beta^2(4\nu c\beta b^2 + 2\nu c\beta + \varepsilon\nu s_0 - 4b^2\varepsilon s_0),\\
K_3 &= -2\beta^2(4\nu c\varepsilon\beta^2 + 8\nu c\varepsilon\beta^2b^2 + 2\lambda r_0\beta + 4\nu\eta b^2\beta + 2\lambda s_0\beta - 2r_0\beta\\
&\quad +4s_0\beta + 2\beta ns_0 + 4n\beta b^2s_0 + 4\lambda s_0b^2\beta + 2\nu\eta\beta - 4\beta b^2s_0 + 4\lambda r_0b^2\beta\\
&\quad -\nu r_{00} - nb^2r_{00} + b^2r_{00}),\\
K_2 &= -4\beta^4(-\nu c\beta + 5\varepsilon s_0 + \varepsilon ns_0 + 2\varepsilon nb^2s_0 - 2b^2\varepsilon s_0),\\
K_1 &= \beta^4(4\nu c\varepsilon\beta^2 + 4\beta ns_0 + 2\lambda\beta s_0 + 2\lambda\beta r_0 - 12s_0\beta + 2\nu\eta\beta\\
&\quad +3r_{00} - nr_{00}),\\
K_0 &= 4(n-3)\varepsilon\beta^6s_0,\ M_7 = 2\nu c(1+2b^2)^2,\\
M_6 &= 2(1+2b^2)(2\nu\eta b^2 + 4\nu c\varepsilon\beta b^2 + 2\lambda b^2r_0 + 2\lambda b^2s_0 - 2r_0 + \lambda r_0\\
&\quad +\varepsilon^2\nu s_0 - 2s_0 + 2\nu c\varepsilon\beta + \nu\eta + \lambda s_0),
\end{aligned}$$

$$
\begin{aligned}
M_5 &= 2\lambda\varepsilon\beta r_0 + 8\lambda b^2\varepsilon\beta s_0 + 2\lambda\varepsilon\beta s_0 - 8\varepsilon\beta b^2 r_0 + 8\lambda\varepsilon\beta b^4 r_0 + 2\nu c\varepsilon^2\beta^2 \\
&\quad +8\lambda\varepsilon\beta b^4 s_0 + 8\lambda\varepsilon\beta b^2 r_0 - \varepsilon\nu r_{00} + 8\nu\eta\varepsilon b^2\beta - 8\nu c b^2\beta^2 \\
&\quad +2\nu\eta\varepsilon\beta - 4\varepsilon\beta r_0 + 8\nu c\varepsilon^2 b^4\beta^2 + 8\nu\eta\varepsilon b^4\beta - 4\nu c\beta^2 \\
&\quad +8\nu c\varepsilon^2 b^2\beta^2 - 2\varepsilon n b^2 r_{00} - 4\varepsilon\beta s_0 - 2b^2\varepsilon r_{00}, \\
M_4 &= -\beta(8\nu\varepsilon c\beta^2 + 16\nu\varepsilon c b^2\beta^2 + 4nb^2\beta s_0 + 4\lambda\beta r_0 + 2\beta\varepsilon^2 s_0 \\
&\quad -2\beta s_0 - 4b^2\beta s_0 + 4\nu\eta\beta - 4r_0\beta + 8\nu\eta b^2\beta + 2n\varepsilon^2\beta s_0 - 8\varepsilon^2 b^2\beta s_0 \\
&\quad +2n\beta s_0 + 8\lambda b^2\beta s_0 + 4\lambda\beta s_0 + 8\lambda b^2\beta r_0 + 2b^2 r_{00} - 2nb^2 r_{00} - \nu r_{00}), \\
M_3 &= -2\beta^2(2\nu\varepsilon^2 c\beta^2 + 4\nu\varepsilon^2 c b^2\beta^2 - \nu c\beta^2 + 4\lambda\varepsilon b^2\beta s_0 + 4\varepsilon\beta s_0 - 2\varepsilon\beta r_0 \\
&\quad +2\lambda\varepsilon\beta r_0 - 4\varepsilon b^2\beta s_0 + 4\nu\eta\varepsilon b^2\beta + 2\lambda\varepsilon\beta s_0 + 2\varepsilon n\beta s_0 + 2\nu\eta\varepsilon\beta \\
&\quad +4\varepsilon n b^2\beta s_0 + 4\lambda\varepsilon b^2\beta r_0 - \varepsilon\nu r_{00} - \varepsilon n b^2 r_{00} + \varepsilon b^2 r_{00}), \\
M_2 &= -\beta^3(-4\nu c\varepsilon\beta^2 + 4\beta n\varepsilon^2 b^2 s_0 + 10\varepsilon^2\beta s_0 - 2\lambda\beta r_0 - 4\varepsilon^2 b^2\beta s_0 \\
&\quad -2n\beta s_0 - 2\nu\eta\beta + 6\beta s_0 - 2\lambda\beta s_0 + 2n\varepsilon^2\beta s_0 - 3r_{00} + nr_{00}), \\
M_1 &= \varepsilon\beta^4(2\nu c\varepsilon\beta^2 + 2\lambda\beta r_0 + 2\lambda\beta s_0 - 12\beta s_0 + 4n\beta s_0 + 2\nu\eta\beta \\
&\quad\ \ +3r_{00} - nr_{00}), \\
M_0 &= 2\varepsilon^2(n-3)\beta^6 s_0,\ \nu = n+1.
\end{aligned}
$$

Replacing y^i in (3.2) by $-y^i$, we get the following

$$
\begin{aligned}
&(-J_6\alpha^6 + J_5\alpha^5 - J_4\alpha^4 + J_3\alpha^3 - J_2\alpha^2 + J_1\alpha - J_0)\arctan^2(\frac{\beta}{\alpha}) \\
&+(K_6\alpha^6 - K_5\alpha^5 + K_4\alpha^4 - K_3\alpha^3 + K_2\alpha^2 - K_1\alpha + K_0)\arctan(\frac{\beta}{\alpha}) \\
&+M_7\alpha^7 - M_6\alpha^6 + M_5\alpha^5 - M_4\alpha^4 + M_3\alpha^3 \\
&-M_2\alpha^2 + M_1\alpha - M_0 = 0.
\end{aligned} \tag{3.3}
$$

(3.2) + (3.3) yields

$$
\begin{aligned}
&(J_5\alpha^5 + J_3\alpha^3 + J_1\alpha)\arctan^2(\frac{\beta}{\alpha}) + M_7\alpha^7 + M_5\alpha^5 + M_3\alpha^3 + M_1\alpha \\
&+(K_6\alpha^6 + K_4\alpha^4 + K_2\alpha^2 + K_0)\arctan(\frac{\beta}{\alpha}) = 0.
\end{aligned} \tag{3.4}
$$

(3.2) − (3.3) yields

$$
\begin{aligned}
&(J_6\alpha^6 + J_4\alpha^4 + J_2\alpha^2 + J_0)\arctan^2(\frac{\beta}{\alpha}) + (K_5\alpha^5 + K_3\alpha^3 + K_1\alpha)\arctan(\frac{\beta}{\alpha}) \\
&= -M_6\alpha^6 - M_4\alpha^4 - M_2\alpha^2 - M_0.
\end{aligned} \tag{3.5}
$$

Using Taylor expansion of $\arctan(\frac{\beta}{\alpha})$, we can find that the right side of (3.5) is an integral expression in y and the left side of (3.5) is a fraction expression in y, so that we get

$$
(J_6\alpha^6 + J_4\alpha^4 + J_2\alpha^2 + J_0)\arctan(\frac{\beta}{\alpha}) + K_5\alpha^5 + K_3\alpha^3 + K_1\alpha = 0, \tag{3.6}
$$

$$
M_6\alpha^6 + M_4\alpha^4 + M_2\alpha^2 + M_0 = 0. \tag{3.7}
$$

Similarly, from (3.6), we get the following

$$J_6\alpha^6 + J_4\alpha^4 + J_2\alpha^2 + J_0 = 0, \tag{3.8}$$

$$K_5\alpha^4 + K_3\alpha^2 + K_1 = 0. \tag{3.9}$$

For the same reason, by (3.4), we have

$$J_5\alpha^4 + J_3\alpha^2 + J_1 = 0, \tag{3.10}$$

$$K_6\alpha^6 + K_4\alpha^4 + K_2\alpha^2 + K_0 = 0, \tag{3.11}$$

$$M_7\alpha^6 + M_5\alpha^4 + M_3\alpha^2 + M_1 = 0. \tag{3.12}$$

(3.10) tells us that $J_1 = 2(n+1)c\beta^6$ has the factor α^2. Because β^6 and α^2 are relatively prime polynomials of (y^i), we immediately obtain $c = 0$.

Now we split the proof into four cases:

(i) $\varepsilon \neq 0$ and $n = 3$;

(ii) $\varepsilon \neq 0$ and $n > 3$;

(iii) $\varepsilon = 0$ and $n > 3$;

(iv) $\varepsilon = 0$ and $n = 3$.

Case i $\varepsilon \neq 0$ and $n = 3$.

In this case, $K_0 = 4(n-3)\varepsilon\beta^6 s_0 = 0$. Hence, (3.11) implies that $K_2 = -16\varepsilon\beta^4 s_0(b^2+2)$ has the factor α^2. This implies $s_0 = 0$. By use of $s_0 = 0$ and $c = 0$, we have $M_7 = M_0 = 0$.

By (3.7), we obtain the following

$$\begin{aligned}&2(1+2b^2)(8\eta b^2 + 2\lambda b^2 r_0 + \lambda r_0 + 4\eta - 2r_0)\alpha^4 - \beta(16\eta\beta + 32\eta b^2\beta + 8\lambda r_0 b^2\beta + 4\lambda r_0\beta\\&-4r_0\beta - 4b^2 r_{00} - 4r_{00})\alpha^2 + \beta^3(8\eta\beta + 2\lambda r_0\beta) = 0.\end{aligned} \tag{3.13}$$

By (3.12), we obtain the following

$$\begin{aligned}&2\varepsilon(1+2b^2)(2\lambda r_0 b^2\beta + 8\eta b^2\beta + \lambda r_0\beta + 4\eta\beta - 2r_0\beta - 2r_{00})\alpha^4 - 4\varepsilon\beta^2(2\lambda r_0 b^2\beta\\&-r_0\beta + \lambda r_0\beta + 8\eta b^2\beta + 4\eta\beta - b^2 r_{00} - 2r_{00})\alpha^2 + \varepsilon\beta^4(2\lambda r_0\beta + 8\eta\beta) = 0.\end{aligned} \tag{3.14}$$

(3.14)$-$(3.13)$\times\varepsilon\beta$ gives

$$4\varepsilon[(1+2b^2)\alpha^2 - \beta^2]r_{00} = 0. \tag{3.15}$$

Because F is non-Riemannian, $(1+2b^2)\alpha^2 - \beta^2 \neq 0$, thus we get

$$r_{00} = 0, \quad r_0 = 0. \tag{3.16}$$

Plugging (3.16) into (2.7) yields $\mathbf{S} = 0$. In this case $\eta = 0$.

Case ii $\varepsilon \neq 0$ and $n > 3$.

From (3.7), we can see that $M_0 = 2\varepsilon^2(n-3)\beta^6 s_0$ has the factor α^2. Since $\varepsilon \neq 0$ and $n > 3$, we have $s_0 = 0$. By use of (3.7) and (3.12) and using the same skills in case (i), we obtain

$$r_{00} = 0, \quad r_0 = 0, \quad \eta = 0, \quad \mathbf{S} = 0. \tag{3.17}$$

Case iii $\varepsilon = 0$ and $n > 3$.

By (3.8), we can see that $J_0 = 2s_0\beta^6(n-3)$ has the factor α^2. Obviously, we can get $s_0 = 0$.

From (3.7), we get the following

$$
\begin{aligned}
&2(1+2b^2)(2(n+1)\eta b^2 + 2\lambda b^2 r_0 + \lambda r_0 + (n+1)\eta - 2r_0)\alpha^4 \\
&-\beta(4(n+1)\eta\beta + 8(n+1)\eta b^2\beta + 8\lambda r_0 b^2\beta + 4\lambda r_0\beta - 4r_0\beta - 2nb^2 r_{00} \\
&-(n+1)r_{00} + 2b^2 r_{00})\alpha^2 + \beta^3(2(n+1)\eta\beta + 2\lambda r_0\beta - nr_{00} + 3r_{00}) = 0.
\end{aligned} \tag{3.18}
$$

From (3.9), we have

$$
\begin{aligned}
&(1+2b^2)(4(n+1)\eta b^2\beta + 4\lambda r_0 b^2\beta - (n+1)r_{00} - 4r_0\beta + 2(n+1)\eta\beta + 2\lambda r_0\beta)\alpha^4 \\
&-2\beta^2(4(n+1)\eta b^2\beta + 2\lambda r_0\beta + 2(n+1)\eta\beta + 4\lambda r_0 b^2\beta - 2r_0\beta - (n+1)r_{00} \\
&-nb^2 r_{00} + b^2 r_{00})\alpha^2 + \beta^4(2\lambda r_0\beta + 2(n+1)\eta\beta - nr_{00} + 3r_{00}) = 0.
\end{aligned} \tag{3.19}
$$

(3.19)−(3.18)×β yields

$$(n+1)[(1+2b^2)\alpha^2 - \beta^2]r_{00} = 0. \tag{3.20}$$

This implies $r_{00} = 0$. For the same reason, we have

$$r_0 = 0, \quad \eta = 0, \quad \mathbf{S} = 0. \tag{3.21}$$

Case iv $\varepsilon = 0$ and $n = 3$.

In this case, $J_0 = 2s_0\beta^6(n-3) = 0$. (3.8) becomes

$$[(1+2b^2)\alpha^4 + (b^2-1)\beta^2\alpha^2 - (b^2+2)\beta^4]s_0 = 0. \tag{3.22}$$

We assert that $s_0 = 0$. Or else, (3.22) tells us that $\beta^4(b^2+2)$ has the factor α^2. This implies $\beta = 0$, but it is impossible by the assumptions. By using the same methods as case iii, we get that (3.21) holds.

Anyway, we obtain $r_{00} = 0$, $s_0 = 0$, $\eta = 0$, $\mathbf{S} = 0$. Which implies that F is of isotropic S-curvature with $c = 0$.

Step 3 (b) $\Rightarrow$ (c) The proof has been contained in Step 2.

Step 4 (c) $\Rightarrow$(d) When $r_{00} = 0$ and $s_0 = 0$, by (2.7), we have $\mathbf{S} = 0$.

Step 5 (d) $\Rightarrow$ (e) $\mathbf{S} = 0$ implies that F is of isotropic S-curvature with $c = 0$. Thus, we obtain $\mathbf{E} = 0$ by the equivalence of (a) and (b).

Step 6 (e) $\Rightarrow$ (a) $\mathbf{E} = 0$ is equivalent to that F is of isotropic mean Berwald curvature with $c = 0$, that is, (b) holds with $c = 0$. By the equivalence of (a) and (b), we know that F has isotropic S-curvature with $c = 0$. This completes the proof. Theorem 1.1 is proved completely.

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关于一类弱Berwald 的 (α, β) -度量

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摘 要: 本文研究了一类重要的形如 $F = \alpha + \varepsilon\beta + \beta \ \arctan(\frac{\beta}{\alpha})$ (ε为常数)的弱Berwald (α, β) -度量. 利用S -曲率公式, 获得了这类度量为弱Berwald度量的充要条件. 并且还证明了F 为具有标量旗曲率的弱Berwald度量当且仅当它们为Berwald 度量且旗曲率消失.

关键词: (α, β) -度量; 弱Berwald度量; 旗曲率

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