Vol. 34 (2014)
No. 4

数 学 杂 志
J. of Math. (PRC)

NEW INEQUALITIES AND CHARACTERIZATIONS FOR L_p-BALLS

ZHENG Lv-zhou[1], WEI Zheng-li[2]

(1. *College of Mathematics and statistics, Hubei Normal University, Huangshi 435002, China*)
(2. *Department of Mathematics, Shanghai University, Shanghai 200444, China*)

Abstract: In this paper, we research some related problems of L_p-ball and obtain several new characterizations and inequalities for L_p-balls by using dual mixed volumes, spherical Radon transform and Fourier transform. One of the inequalities is related to the famous maximal slice conjecture.

Keywords: convex body; dual mixed volume; L_p centroid body; L_p-ball

2010 MR Subject Classification: 52A20

Document code: **A** **Article ID:** **0255-7797(2014)04-0617-10**

1 Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space. A convex body $K \subseteq \mathbb{R}^n$ is a compact convex subset with non-empty interior. Associated with a compact convex set K is its support function h_K defined on $\mathbb{R}^n$ by $h_K(x) = \max\{\langle x,\ y\rangle : y \in K\}$, where $\langle x,\ y\rangle$ is the usual inner product of x and y in $\mathbb{R}^n$. The support function h_K is positively homogeneous of degree 1. We shall usually be concerned with the restriction of the support function to the unit sphere S^{n-1} in $\mathbb{R}^n$.

For a compact set L in $\mathbb{R}^n$ which is star shaped with respect to the origin o, define the radial function ρ_L of L by $\rho_L(x) = \max\{\lambda \geq 0 : \lambda x \in L\}$, $x \in \mathbb{R}^n - \{o\}$. The radial function is positively homogeneous of degree -1. One can identify the radial function with its restriction to the unit sphere S^{n-1} in $\mathbb{R}^n$.

If ρ_L is continuous, we shall call L a star body. A star body which is centrally symmetric with respect to the origin will be called a centered body. We shall use $\mathscr{S}$ and $\mathscr{S}_e$ to denote the set of star bodies and the set of centered bodies, respectively.

For a convex body K containing the origin in its interior, the polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : \langle x,\ y\rangle \leq 1, y \in K\}.$$

It is easy to verify that $K^{**} = K$, and that $h_{K^*} = \rho_K^{-1},\ \ \rho_{K^*} = h_K^{-1}$.

* **Received date:** 2012-03-05 **Accepted date:** 2012-06-20

Foundation item: Supported by NSFC (11001163); Innovation Program of Shanghai Municipal Education Commission (11YZ11).

Biography: Zheng lvzhou(1967–), female, born in Jingshan, Hubei, associate professor, major in differential equation and convex geometry.

If K is a centered convex body, then the reciprocal of its radial function induces a norm on $\mathbb{R}^n$, denoted by $\|\cdot\|_K$, whose unit ball is K. That is,

$$\|x\|_K = \rho_K(x)^{-1},\ x \in \mathbb{R}^n.$$

Conversely, if $(\mathbb{R}^n,\ \|\cdot\|)$ is a normed space with unit ball K (i.e., $K = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$), it is easily seen that $\|\cdot\| = \|\cdot\|_K$.

A centered convex body K in $\mathbb{R}^n$ is called an L_p-ball, if it is the unit ball of an n-dimensional subspace of some L_p-space. Denote by $\mathcal{L}_p$ the class of L_p-balls. It is noted that the class $\mathcal{L}_p$ is well known and is important in the local theory of Banach spaces. See, for example, [11, 12].

Note that $\mathcal{L}_2$ is the class of centered ellipsoids. The most important example of an L_p-ball is the unit ball of space l_p^n given by

$$\{x = (x_1, \cdots, x_n) \in \mathbb{R}^n :| x_1 |^p + \cdots + | x_n |^p \leq 1\}.$$

Recall that the L_p-centroid body $\Gamma_p L$ of a star body L is defined by

$$h_{\Gamma_p L}(u)^p = \frac{1}{V(L)} \int_L | \langle u, x\rangle |^p \ \ dx, u \in S^{n-1}, p \geq 1.$$

Therefore the polar body $\Gamma_p^* L$ of $\Gamma_p L$ belongs to $\mathcal{L}_p$. The first-order centroid body $\Gamma_1 L$ is usually called the centroid body of L [23] and is denoted by ΓL. For $p = 2$, they are the Legendre ellipsoids which appears in classical mechanics. In the important paper [16], the L_p analogs of two famous affine isoperimetric inequalities in convex geometry, which are the Busemann-Petty centroid inequality and Petty projection inequality, are established.

Using polar coordinates, one can write the last integral as an integral over S^{n-1},

$$h_{\Gamma_p L}(u)^p = \frac{1}{(n+p)V(L)} \int_{S^{n-1}} | \langle u, v\rangle |^p \ \rho_L(v)^{n+p} dv.$$

In this paper, we establish several new characterizations and inequalities for L_p-balls by using dual mixed volumes, spherical Radon transform and Fourier transform. One of the inequalities is related to the famous maximal slice conjecture.

To make the paper self-contained, we will recall some basic facts on dual mixed volume and Radon transform. For more details, one can refer [2, 4, 9, 13–16, 25, 26], etc.

2 Dual Mixed Volume and Radon Transform

As usual, S^{n-1} denotes the unit sphere, B_n the unit ball and o the origin in the n-dimensional Euclidean space $\mathbb{R}^n$. The surface area of the unit sphere S^{n-1} and the volume of the unit ball B_n in $\mathbb{R}^n$ are denoted by α_{n-1} and ω_n, respectively. Note that $\alpha_{n-1} = n\omega_n$, and $\omega_n = \frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}$. For real $p \geq 1$, define $c_{n,p}$ by $c_{n,p} = \frac{\omega_{n+p}}{\omega_2 \omega_n \omega_{p-1}}$.

For $K_1, \cdots, K_r \in \mathscr{S}$, $\lambda_1, \cdots, \lambda_r \geq 0$, the radial linear combination $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r \in \mathscr{S}$, is defined by

$$\rho_{\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r} = \lambda_1 \rho_{K_1} + \cdots + \lambda_r \rho_{K_r}.$$

The volume of the radial linear combination $\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i,

$$V(\lambda_1 K_1 \widetilde{+} \cdots \widetilde{+} \lambda_r K_r) = \sum \widetilde{V}(K_{i_1}, \cdots, K_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n},$$

where the sum is taken over all n-tuples $(i_1, \cdots, i_n)$ whose entries are positive integers not exceeding r. The coefficient $\widetilde{V}(K_{i_1}, \cdots, K_{i_n})$ is non-negative and depends only on the bodies $K_{i_1}, \cdots, K_{i_n}$. It is called the dual mixed volume of $K_{i_1}, \cdots, K_{i_n}$. One has the following integral representation of dual mixed volumes:

$$\widetilde{V}(K_1, \cdots, K_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{K_1}(u) \cdots \rho_{K_n}(u) du,$$

where du is the volume element of S^{n-1}.

Dual mixed volumes are the counterparts of mixed volumes. While the mixed volumes date back to Minkowski in the last century, the dual mixed volumes were only recently discovered. They play the roles in the study of cross-sections of convex bodies as the mixed volumes do for the study of projections of convex bodies. However, this duality is not at all trivial. One should read the book [2] for an excellent illustration. Dual mixed volumes are far from well understood. Their applications to the characterizations of intersection bodies and the solutions of the Busemann-Petty problem are very recent developments. See [7, 20, 21, 27].

Denote $\widetilde{V}(K, \cdots, K, L, \cdots, L)$ by $\widetilde{V}_i(K, L)$, where K appears $n-i$ times and L appears i times. The dual quermassintegral $\widetilde{W}_{n-i}(K)$ is given by $\widetilde{V}_i(K, B_n)$. The importance of the dual quermassintegrals lies in the fact that the $(n-i)$th dual quermassintegral of a star body K is proportional to the mean of the i-dimensional volumes of the slices of K by the i-dimensional subspaces of $\mathbb{R}^n$, that is

$$\widetilde{W}_{n-i}(K) = \frac{w_n}{w_i} \int_{G(n,\, i)} \mathrm{vol}_i(K \cap \xi) d\mu_i(\xi),$$

where $G(n,\, i)$ is the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$, and μ_i the Haar measure on $G(n, i)$, normalized by $\mu_i(G(n, i)) = 1$. For $K \in \mathscr{S}$, the intersection $K \cap u^\perp$ is a star body in $(n-1)$-dimensional space. Let $\widetilde{W}_{n-1-i}(K \cap u^\perp)$ be the $(n-1-i)$ th dual quermassintegral of $K \cap u^\perp$ in $\mathbb{R}^{n-1}$, which is called the dual $(n-1-i)$-girth of K in the direction u.

A slight extension of the notation $\widetilde{V}_i(K, L)$ is

$$\widetilde{V}_r(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho_K(u)^{n-r} \rho_L(u)^r du, \quad r \in \mathbb{R}.$$

The dual Minkowski inequalities state that

$$\widetilde{V}_r(K, L)^n \le V(K)^{n-r} V(L)^r, \quad \text{if} \quad r > 0,$$

$$\widetilde{V}_r(K, L)^n \ge V(K)^{n-r} V(L)^r, \quad \text{if} \quad r < 0,$$

with equalities if and only if K and L are dilations of each other.

The intersection body IK of a star body $K \in \mathscr{S}$ is defined as the centered body whose radial function is given by

$$\rho_{IK}(u) = \mathrm{vol}_{n-1}(K \cap u^{\perp}), \quad u \in S^{n-1},$$

where $u^{\perp}$ is the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to u. We will follow Zhang [25] to consider some generalizations of this definition.

The intersection body of order i of a star body K, I_iK, is defined by

$$\rho_{I_iK}(u) = \widetilde{W}_{n-1-i}(K \cap u^{\perp}), \ \ u \in S^{n-1}.$$

Hence, $IK = I_{n-1}K$.

The intersection body of order i is closely related to the spherical Radon transform. For $f \in C(S^{n-1})$, the spherical Radon transform of f, Rf , is defined by

$$(Rf)(u) = \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} f(\nu)ds(\nu),$$

where ds is the volume element of $S^{n-1} \cap u^{\perp}$. By applying the spherical Radon transform to the ith power of the radial function of a star body, we have

$$\begin{aligned}(R\rho_K^i)(u) &= \frac{1}{n-1}\int_{S^{n-1}\cap u^{\perp}} \rho_K^i(\nu)ds(\nu)\\ &= \frac{w_{n-1}}{iw_i}\int_{G(n-1,\ i)}\int_{S^{n-2}\cap\xi} \rho_K^i(\nu)dS_{i-1}(B;\ \nu)d\mu_i(\xi)\\ &= \frac{w_{n-1}}{w_i}\int_{G(n-1,\ i)} \mathrm{vol}_i(K\cap u^{\perp})d\mu_i(\xi).\end{aligned}$$

Hence, we have

$$\rho_{I_iK}(u) = (R\rho_K^i)(u) = \widetilde{W}_{n-1-i}(K \cap u^{\perp}).$$

When restricted to $C_e^{\infty}(S^{n-1})$, the spherical Radon transform $R : C_e^{\infty}(S^{n-1}) \to C_e^{\infty}(S^{n-1})$ is a continuous bijection (see Helgason [6, p.161]). It is also selfadjoint, i.e., for $f, g \in C(S^{n-1})$, $(f, Rg) = (Rf, g)$.

For $K \in \mathscr{S}_e$, we call the distribution $R^{-1}\rho_K$ the dual generating distribution of K, denoted by $\widetilde{\mu}_K$.

Let ϕ be a function from the Schwartz space $\mathcal{S}$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. We define the Fourier transform of ϕ by

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(x)e^{-i(x,\ \xi)}dx, \xi \in \mathbb{R}^n.$$

The Fourier transform of a distribution f is defined by $\langle \widehat{f}, \phi\rangle = \langle f, \widehat{\phi}\rangle$ for every test function $\phi \in \mathcal{S}$. We say that a distribution is positively definite if its Fourier transform is a positive distribution, in the sense that $\langle \widehat{f},\ \phi\rangle \geq 0$ for every non-negative test function ϕ.

For the applications of Fourier transform to convex geometry, one can refer the papers, for example, [4, 5, 7, 8, 10, 25] and the book [9].

3 Main Results

A well-known theorem of Lewis [11] will be used in the proof of Theorem 1. We presented it here in the following form proved in [17] (see Theorem 8.2) by Lutwak, Yang and Zhang.

Lemma 1 If ℓ is an n-dimensional subspace of L_p, then ℓ is isometric to the Banach space $(\mathbb{R}^n, \|\cdot\|)$ where the norm $\|\cdot\|$ can be represented by a finite Borel measure, μ, such that for all $x \in \mathbb{R}^n$,

$$\|x\| = (\int_{S^{n-1}} | x \cdot v |^p \, d\mu(v))^{\frac{1}{p}},$$

and

$$| x | = (\int_{S^{n-1}} | x \cdot v |^2 \, d\mu(v))^{\frac{1}{2}}.$$

Theorem 1 Let $K \in \mathcal{L}_p$, $1 \leq p < \infty$, and L be a star body. Then

$$\widetilde{V}_{-p}(L,\ K) = \frac{(n+p)V(L)}{n} \int_{S^{n-1}} h^p_{\Gamma_p L}(v) d\mu_K(v),$$

where μ_K is a finite Borel measure on S^{n-1}. Specially, if L is the unit ball B_n, then

$$\widetilde{V}_{-p}(B_n,\ K) = n\omega_n c_{n-2,\ p}.$$

Proof According to Lemma 1, we know that if $(\mathbb{R}^n, \|\cdot\|_K)$ is a subspace of L_p, $p \geq 1$, then there exists a position of the body K and a finite Borel measure μ_K on S^{n-1} such that for all $x \in \mathbb{R}^n$,

$$\|x\|^p_K = \int_{S^{n-1}} | x \cdot v |^p \, d\mu_K(v),$$

and

$$| x |^2 = \int_{S^{n-1}} | x \cdot v |^2 \, d\mu_K(v). \tag{$*$}$$

From the definition of dual mixed volume and Fubini's theorem, we have

$$\begin{aligned}
\widetilde{V}_{-p}(L,\ K) &= \frac{1}{n} \int_{S^{n-1}} \rho_L^{n+p}(u) \rho_K^{-p}(u) du \\
&= \frac{1}{n} \int_{S^{n-1}} \rho_L^{n+p}(u) (\int_{S^{n-1}} | u \cdot v |^p \, d\mu_K(v)) du \\
&= \frac{1}{n} \int_{S^{n-1}} (\int_{S^{n-1}} \rho_L^{n+p}(u) \, | u \cdot v |^p \, du) d\mu_K(v) \\
&= \frac{(n+p)V(L)}{n} \int_{S^{n-1}} (\frac{1}{(n+p)V(L)} \int_{S^{n-1}} \rho_L^{n+p}(u) \, | u \cdot v |^p \, du) d\mu_K(v) \\
&= \frac{(n+p)V(L)}{n} \int_{S^{n-1}} h^p_{\Gamma_p L}(v) d\mu_K(v).
\end{aligned}$$

When L is the unit ball B_n,

$$h^p_{\Gamma_p B_n}(v) = \frac{1}{(n+p)\omega_n}\int_{S^{n-1}} \rho^{n+p}_{B_n}(u) \mid u \centerdot v \mid^p du = \frac{1}{(n+p)\omega_n}\int_{S^{n-1}} \mid u \centerdot v \mid^p du = \frac{n\omega_n c_{n-2,p}}{(n+p)\omega_n}.$$

Finally, we only need to compute the measure of S^{n-1} with respect to μ_K. Let σ be the normalized Haar measure on the sphere. Integrating equation (*) with respect to σ we get

$$\begin{aligned} 1 &= \int_{S^{n-1}} \mid x \mid^2 d\sigma(x) = \int_{S^{n-1}}\int_{S^{n-1}} \mid x \centerdot v \mid^2 d\mu_K(v) d\sigma(x) \\ &= \int_{S^{n-1}} \mid x_1 \mid^2 d\sigma(x) \cdot \int_{S^{n-1}} d\mu_K(v) = \frac{1}{n}\int_{S^{n-1}} d\mu_K(v). \end{aligned}$$

Hence, when $L = B_n$, we have $\widetilde{V}_{-p}(B_n,\ K) = n\omega_n c_{n-2,\ p}$.

This completes the proof.

From Theorem 1, we can get the following property immediately.

Corollary 1 Let $K \in \mathcal{L}_p$, $1 \le p < \infty$. Then

$$\int_{S^{n-1}} h^p_{\Gamma_p K}(v) d\mu_K(v) = \frac{n}{n+p},$$

where μ_K is a finite Borel measure on S^{n-1}.

Theorem 2 Let $K \in \mathcal{L}_p$, $1 \le p < \infty$. Then for any star bodies L and M, we have

$$\Gamma_p L \subseteq \Gamma_p M \Longrightarrow \frac{\widetilde{V}_{-p}(L,\ K)}{V(L)} \le \frac{\widetilde{V}_{-p}(M,\ K)}{V(M)}.$$

Proof From the proof of Theorem 1, we have

$$\begin{aligned} \frac{\widetilde{V}_{-p}(L,\ K)}{V(L)} &= \frac{n+p}{n}\int_{S^{n-1}} h^p_{\Gamma_p L}(v) d\mu_K(v), \\ \frac{\widetilde{V}_{-p}(M,\ K)}{V(M)} &= \frac{n+p}{n}\int_{S^{n-1}} h^p_{\Gamma_p M}(v) d\mu_K(v). \end{aligned}$$

If $\Gamma_p L \subseteq \Gamma_p M$, then we have $h_{\Gamma_p L} \le h_{\Gamma_p M}$, Therefore, the following inequality

$$\frac{\widetilde{V}_{-p}(L,\ K)}{V(L)} \le \frac{\widetilde{V}_{-p}(M,\ K)}{V(M)}$$

holds.

This completes the proof.

Corollary 2 Let $K \in \mathcal{L}_p$, $1 \le p < \infty$. Then for any star body L, we have

$$\Gamma_p L \subseteq \Gamma_p K \Longrightarrow V(L) \le V(K).$$

Proof Let $M = K$ in Theorem 2. Then $\frac{\widetilde{V}_{-p}(L,\ K)}{V(L)} \le 1$. By the dual Minkowski inequality, we have

$$\frac{\widetilde{V}_{-p}(L,\ K)}{V(L)} \ge \frac{V(L)^{\frac{n+p}{n}} V(K)^{\frac{-p}{n}}}{V(L)}.$$

Therefore, $V(L) \le V(K)$.

This completes the proof.

Remark In [5] (see Theorem 4.11 and Corollary 4.13), the authors proved the similar results as our Theorem 2 and Corollary 2, respectively, when L and M are centered star bodies.

Theorem 3 Let $K \in \mathscr{S}_e$. If K is an intersection body, then for all L_1, L_2, M_1, $M_2 \in \mathscr{S}_e$,

$$\frac{\widetilde{V}(L_1 \widetilde{+} L_2,\ i;\ K,\ 1;\ B_n)}{\widetilde{V}(M_1 \widetilde{+} M_2,\ i;\ K,\ 1;\ B_n)} \le \sum_{k=0}^{i} \max_{u\in S^{n-1}} \frac{\widetilde{V}(L_1\cap u^\perp,\ k;\ L_2\cap u^\perp,\ i-k;\ B_n,\ n-i)}{\widetilde{V}(L_1\cap u^\perp,\ k;\ L_2\cap u^\perp,\ i-k;\ B_n,\ n-i)}.$$

Proof Suppose K is an intersection body with dual generating measure $\widetilde{\mu_k}$. Then

$$\begin{aligned}
\frac{\widetilde{V}(L_1 \widetilde{+} L_2,\ i;\ K,\ 1;\ B_n)}{\widetilde{V}(M_1 \widetilde{+} M_2,\ i;\ K,\ 1;\ B_n)} &= \frac{(\rho^i_{L_1\widetilde{+}L_2},\ \rho_k)}{(\rho^i_{M_1\widetilde{+}M_2},\ \rho_k)} = \frac{((\rho_{L_1}+\rho_{L_2})^i,\ \rho_K)}{((\rho_{M_1}+\rho_{M_2})^i,\ \rho_K)} = \frac{(R(\rho_{L_1}+\rho_{L_2})^i,\ R^{-1}\rho_K)}{(R(\rho_{M_1}+\rho_{M_2})^i,\ R^{-1}\rho_K)}\\
&= \frac{(R\sum_{k=0}^{i}\binom{i}{k}\rho^k_{L_1}\rho^{i-k}_{L_2},\ R^{-1}\rho_K)}{(R\sum_{k=0}^{i}\binom{i}{k}\rho^k_{M_1}\rho^{i-k}_{M_2},\ R^{-1}\rho_K)}\\
&= \frac{\sum_{k=0}^{i}\binom{i}{k}\int_{S^{n-1}} R(\rho^k_{L_1}\rho^{i-k}_{L_2})d\widetilde{u_k}(u)}{\sum_{k=0}^{i}\binom{i}{k}\int_{S^{n-1}} R(\rho^k_{M_1}\rho^{i-k}_{M_2})d\widetilde{u_k}(u)}\\
&\le \sum_{k=0}^{i}\max_{u\in S^{n-1}} \frac{R(\rho^k_{L_1}\rho^{i-k}_{L_2})}{R(\rho^k_{M_1}\rho^{i-k}_{M_2})}\\
&= \sum_{k=0}^{i}\max_{u\in S^{n-1}} \frac{\int_{S^{n-1}\cap u^\perp}\rho^k_{L_1}\rho^{i-k}_{L_2}ds(\nu)}{\int_{S^{n-1}\cap u^\perp}\rho^k_{M_1}\rho^{i-k}_{M_2}ds(\nu)}\\
&= \sum_{k=0}^{i}\max_{u\in S^{n-1}} \frac{\int_{S^{n-1}}\rho^k_{L_1\cap u^\perp}\rho^{i-k}_{L_2\cap u^\perp}ds(\nu)}{\int_{S^{n-1}}\rho^k_{M_1\cap u^\perp}\rho^{i-k}_{M_2\cap u^\perp}ds(\nu)}\\
&= \sum_{k=0}^{i}\max_{u\in S^{n-1}} \frac{\widetilde{V}(L_1\cap u^\perp,\ k;\ L_2\cap u^\perp,\ i-k;\ B_n,\ n-i)}{\widetilde{V}(M_1\cap u^\perp,\ k;\ M_2\cap u^\perp,\ i-k;\ B_n,\ n-i)}.
\end{aligned}$$

This completes the proof.

Corollary 3 Let $K \in \mathscr{S}_e$. If K is an intersection body. Then for all L, $M \in \mathscr{S}_e$,

$$\frac{\widetilde{V}(L,\ i;\ K,\ 1;\ B)}{\widetilde{V}(M,\ i;\ K,\ 1;\ B)} \le \max_{u\in S^{n-1}} \frac{\widetilde{W}_{n-1-i}(L\cap u^\perp)}{\widetilde{W}_{n-1-i}(M\cap u^\perp)}.$$

Proof From Theorem 3, we have

$$\begin{aligned}
&\frac{\widetilde{V}(L,\ i;\ K,\ 1;\ B)}{\widetilde{V}(M,\ i;\ K,\ 1;\ B)} = \frac{(\rho^i_L,\ \rho_K)}{(\rho^i_M,\ \rho_K)} = \frac{(R\rho^i_L,\ R^{-1}\rho_K)}{(R\rho^i_M,\ R^{-1}\rho_K)}\\
=&\frac{\int_{u\in S^{n-1}}\rho_{I_iL}(u)d\widetilde{u_k}(u)}{\int_{u\in S^{n-1}}\rho_{I_iM}(u)d\widetilde{u_k}(u)} \le \max_{u\in S^{n-1}}\frac{\rho_{I_iL}(u)}{\rho_{I_iL}(u)} = \max_{u\in S^{n-1}}\frac{\widetilde{W}_{n-1-i}(L\cap u^\perp)}{\widetilde{W}_{n-1-i}(M\cap u^\perp)}.
\end{aligned}$$

This completes the proof.

To prove the Theorem 4, we will need the following version of Parseval's formula on the sphere proved in [7].

Lemma 2 If K and L are origin symmetric infinitely smooth convex bodies in $\mathbb{R}^n$ and $0<p<n$, then $(\|x\|_K^{-p})^\wedge$ and $(\|x\|_L^{-n+p})^\wedge$ are continuous functions on S^{n-1} and

$$\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\xi)(\|x\|_L^{-n+p})^\wedge(\xi)d\xi=(2\pi)^n\int_{S^{n-1}}\|x\|_K^{-p}\|x\|_L^{-n+p}dx.$$

The concept of embedding of a normed spaces in L_{-p} with $0<p<n$ was introduced in [8] by Koldobsky. It was also proved that, as for positive p, there is a Fourier analytic characterization for such embeddings, namely a space $(\mathbb{R}^n,\|\centerdot\|)$ embeds in L_{-p} if and only if the Fourier transform of $\|\centerdot\|^{-p}$ is a positive distribution in $\mathbb{R}^n$. In [10], the unit balls of such spaces are called L_{-p}-balls or p-intersection bodies.

Theorem 4 Let K and L be origin symmetric infinitely smooth convex bodies in $\mathbb{R}^n$. If L is a L_{-p}-ball, $0<p<n$, then $\widetilde{V}_p(K,\ L)\le CV(K)^{\frac{p}{n}}\max\limits_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi)$, where

$$C=\frac{\Gamma(\frac{n-p}{2})}{2^p\pi^{\frac{n}{2}}n^{\frac{n-p}{n}}\Gamma(\frac{p}{2})}\alpha_{n-1}^{\frac{n-p}{n}}.$$

Proof According to the definition of dual mixed volume and Lemma 2, we have

$$\begin{aligned}\widetilde{V}_p(K,\ L)&=\frac{1}{n}\int_{S^{n-1}}\rho_K^p(u)\rho_L^{n-p}(u)du\\&=\frac{1}{n}\int_{S^{n-1}}\|x\|_K^{-p}\|x\|_L^{-n+p}dx\\&=\frac{1}{(2\pi)^n n}\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\xi)(\|x\|_L^{-n+p})^\wedge(\xi)d\xi.\end{aligned}$$

If L is a L_{-p}-ball, then $(\|x\|_K^{-p})^\wedge(\xi)\ge 0$, therefore

$$\widetilde{V}_p(K,\ L)\le\frac{1}{(2\pi)^n n}\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\xi)d\xi\cdot\max_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi)$$

Using that (see [3], p.192):

$$(|\ x\ |_2^{-n+p})^\wedge(\xi)=2^p\pi^{\frac{n}{2}}\frac{\Gamma(\frac{p}{2})}{\Gamma(\frac{n-p}{2})}\ |\ \xi\ |_2^{-p},$$

and applying Parseval's formula again, then Hölder's inequality, we have

$$\begin{aligned}\widetilde{V}_p(K,\ L)&\le\frac{2^{-p}\pi^{\frac{-n}{2}}}{(2\pi)^n n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\xi)(|\ x\ |_2^{-n+p})^\wedge(\xi)d\xi\times\max_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi)\\&=\frac{2^{-p}\pi^{\frac{-n}{2}}}{n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}\int_{S^{n-1}}\|x\|_K^{-p}dx\cdot\max_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi)\\&\le\frac{2^{-p}\pi^{\frac{-n}{2}}}{n}\frac{\Gamma(\frac{n-p}{2})}{\Gamma(\frac{p}{2})}(\int_{S^{n-1}}\|x\|_K^{-n}dx)^{\frac{p}{n}}\cdot(\alpha_{n-1})^{\frac{n-p}{n}}\cdot\max_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi)\\&=CV(K)^{\frac{p}{n}}\cdot\max_{\xi\in S^{n-1}}(\|x\|_L^{-n+p})^\wedge(\xi),\end{aligned}$$

where

$$C = \frac{\Gamma(\frac{n-p}{2})}{2^p \pi^{\frac{n}{2}} n^{\frac{n-p}{n}} \Gamma(\frac{p}{2})} \alpha_{n-1}^{\frac{n-p}{n}}.$$

This completes the proof.

When $K = L$, we can get the following result from Theorem 4 immediately.

Corollary 5 Let K be an origin symmetric infinitely smooth convex bodies in $\mathbb{R}^n$. If K is a L_{-p}-ball, $0 < p < n$, then

$$V(K)^{\frac{n-p}{n}} \leq C \max_{\xi \in S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\xi),$$

where

$$C = \frac{\Gamma(\frac{n-p}{2})}{2^p \pi^{\frac{n}{2}} n^{\frac{n-p}{n}} \Gamma(\frac{p}{2})} \alpha_{n-1}^{\frac{n-p}{n}}.$$

From Theorem 3.8 in [9], we have

$$(\|x\|_L^{-n+1})^\wedge(\xi) = \pi(n-1)\mathrm{Vol}_{n-1}(K \cap \xi^\perp).$$

So according to Corollary 5, when $p = 1$, we can get

Corollary 6 Let K be an origin symmetric infinitely smooth convex bodies in $\mathbb{R}^n$. If K is a L_{-1}-ball, then

$$V(K)^{\frac{n-1}{n}} \leq \frac{\Gamma(\frac{n-1}{2})}{2^{\frac{1}{n}} \pi (n\Gamma(\frac{n}{2}))^{\frac{n-1}{n}}} \max_{\xi \in S^{n-1}} \mathrm{vol}_{n-1}(K \cap \xi^\perp).$$

Corollary 6 is related to the maximal slice conjecture. This conjecture states that for any origin symmetric convex body there exists an universal constant $c > 0$ such that

$$V(K)^{\frac{n-1}{n}} \leq c \max_{\xi \in S^{n-1}} \mathrm{vol}_{n-1}(K \cap \xi^\perp).$$

For related problems, see [1, 22].

References

[1] Ball K. Cube slicing in $\mathbb{R}^n$[J]. Proc. Amer. Math. Soc., 1986, 97 : 465–473.

[2] Gardner R J. Geometric tomography[M]. Cambridge: Cambridge University Press, 1995.

[3] Gelfand I, Shilov G. Generalized functions, vol.1, properties and operations[J]. New York, London: Academic Press, 1964.

[4] Goodey P, Zhang Gaoyong. Characterizations and inequalities of zonoids[J]. J. London Math. Soc., 1996, 53: 184–196.

[5] Grinberg Eric L, Zhang Gaoyong. Convolutions, transforms, and convex bodies[J]. Proc. London math. Soc., 1999, 78: 77–115.

[6] Helgason S. Groups and geometric analysis[M]. Academic Press, 1984.

[7] Koldobsky A. A generalization of the Busemann-Petty problem on sections of convex bodies[J]. Israel J. Math., 1999, 110: 75–91.

[8] Koldobsky A. Positive definite distributions and subspaces of L_{-p} with applications to stable processes[J]. Canad. Math. Bull., 1999, 42: 344–353.

[9] Koldobsky A. Fourier analysis in convex geometry[M]. Providence RI: American Mathematical Society, 2005.

[10] Koldobsky A, Pajor A, Yaskin V. Inequalities of the Kahane-Khinchin type and sections of L_p-balls[J]. Studia Math, 2008, 184: 217–231.

[11] Lewis D R. Finite dimensional subspaces of L_p[J]. Studia Math., 1978, 63: 207–212.

[12] Lindenstrauss J, Milman V. The local theory of normed spaces and its applications to convexity, handbook of convex geometry[M]. Amsterdam: North-Holland, 1993.

[13] Lutwak E. Centroid bodies and dual mixed volumes[J]. Proc. London Math. Soc., 1990, 69: 365–391.

[14] Lutwak E. The Brunn-Minkowski-Firey theory I: mixed volumes and the Minkowski problem[J]. J. Differential Geom., 1993, 38: 131–150.

[15] Lutwak E. The Brunn-Minkowski-Firey theory II: affine and geominimal surface areas[J]. Advances in Math., 1996, 118: 244–294.

[16] Lutwak E, Deane Yang, Zhang Gaoyong. L_p affine isoperimetric inequalities[J]. J. Differential Geom., 2000, 56: 111–132.

[17] Lutwak E, Deane Yang, Zhang Gaoyong. L_p John ellipsoids[J]. Proc. London Math. Soc, 2005, 90: 497–520.

[18] Lutwak E, Deane Yang, Zhang Gaoyong. Sharpe affine L_p Sobolev inequalities[J]. J. Differential Geom., 2002, 62: 17–38.

[19] Lutwak E, Zhang Gaoyong. Blaschke-Santaló inequalities[J]. J. Differential Geom., 1997, 47: 1–16.

[20] Milman E. Dual mixed volumes and the slicing problem[J]. Adv. Math., 2006, 207: 566–598.

[21] Milman E. Generalized intersection bodies[J]. J. Funct. Anal., 2006, 240: 530–567.

[22] Milman V D, Pajor A. Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed n-dimensional space, geometric aspects of functional analysis[M]. Lecture Notes in Math., Berlin: Springer-Verlag, 1989.

[23] Petty C M. Centroid bodies[J]. Pacific J. Math., 1961, 11: 1535–1547.

[24] Schneider R. Convex bodies: the Brunn-Minkowski theory[M]. Cambridge: Cambridge University Press, 1993.

[25] Zhang Gaoyong. Centered bodies and dual mixed volumes[J]. Trans. Amer. Math. Soc., 1994, 345: 777–801.

[26] Zhang Gaoyong. Dual kinematic formulas[J]. Trans. Amer. Math. Soc., 1999, 351: 985–995.

[27] Zhang Gaoyong. A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$[J]. Annals of Math., 1999, 149: 535–543.

关于 L_p 球的几个新不等式和性质

郑绿洲[1], 魏正理[2]

(1. 湖北师范学院数学与统计学院, 湖北 黄石 435002)
(2. 上海大学理学院数学系, 上海 200444)

摘要: 本文研究了 L_p 球的相关问题. 利用对偶混合体积、球面 Radon 变换和 Fourier 变换的方法, 获得了关于 L_p 球的几个新不等式和性质, 其中一个不等式与著名的最大切片猜想有关.

关键词: 凸体; 对偶混合体积; L_p 质心体; L_p 球

MR(2010)**主题分类号**: 52A20 **中图分类号**: O186.5