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Existence and Uniqueness of Solutions to Time-delays Stochastic Fractional Differential Equations with Non-Lipschitz Coefficients

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Abstract In this paper, we consider the existence and uniqueness of solutions to time-varying delays stochastic fractional differential equations (SFDEs) with non-Lipschitz coefficients. By using fractional calculus and stochastic analysis, we can obtain the existence result of solutions for stochastic fractional differential equations.

Keywords Existence and uniqueness, Stochastic fractional differential equations, Time-varying delays, Non-Lipschitz coefficients.

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1. Introduction

In recent years, fractional differential equation has been a growing field of research because of their widespread applications in many real life problems, such as image processing, biology, economics, fitting of experimental data and control theory. Classical theory and applications of fractional differential equations are presented in the monographs (see [9, 10, 15, 17, 19]). Stochastic differential equations have become an active area of investigation due to their applications in finance markets, biology, telecommunications networks and other fields [2, 5, 6, 13, 16]. Further, many authors studied the existence, uniqueness, stability and controllability of solutions for stochastic differential equations by using stochastic analysis theory and fixed point theorem in related literature, see [3, 7, 8, 20–22].

In the paper [1], Abouagwa et al. consider stochastic fractional differential equations of Itô-Doob type in the following form:

$$\begin{cases} dx(t) = b\big(t, x(t)\big)dt + \sigma_1\big(t, x(t)\big)dB(t) + \sigma_2\big(t, z(x)\big)(dt)^\alpha, \ t \in [0, T], \\ x(t) = x_0 \in \mathbb{R}^n, \end{cases}$$

where $\frac{1}{2} < \alpha < 1$, T denotes a positive real number, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\sigma_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable continuous functions. $B(t)$ is Brownian motion defined on the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. Approximation properties for solutions to equations were established by fractional calculus, stochastic analysis, elementary inequalities and so on.

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Luo et al. [11] investigate a class of stochastic fractional differential equations with time-delays

$$\begin{cases} dx(t) = b\big(t, x(t), x(t-\tau)\big)dt + \sigma_1\big(t, x(t), x(t-\tau)\big)dB(t) \\ \qquad\qquad + \sigma_2\big(t, x(t), x(t-\tau)\big)(dt)^\alpha,\ t \in J, \\ x(\theta) = \Phi(\theta),\ \theta \in [-\tau, 0] \end{cases}$$

where $J = [0, T]$, $\frac{1}{2} < \alpha < 1$, $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $\sigma_2 : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable continuous functions. $B(t)$ is Brownian motion defined on the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. $\phi : [-\tau, 0] \to \mathbb{R}^n$ is a continuous function satisfying $\mathbb{E}|\phi(\theta)|^2 < \infty$. Under some assumptions, the author obtained an averaging principle for the solution of the considered equations.

Motivated by the discussion above, we are concerned with the following time-varying delays in stochastic fractional differential equations

$$\begin{cases} dz(t) = b\big(t, z(t), z(t-\delta(t))\big)dt + \sigma_1\big(t, z(t), z(t-\delta(t))\big)dB(t) \\ \qquad\qquad + \sigma_2\big(t, z(t), z(t-\delta(t))\big)(dt)^\alpha,\ t \in J, \\ z(t) = \phi(t),\ t \in [-\delta, 0], \end{cases} \tag{1.1}$$

where $\frac{1}{2} < \alpha < 1$, $J = [-\delta, T]$, T denotes a positive real number, $0 \leq \delta(t) \leq \delta$, $b : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma_1 : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $\sigma_2 : J \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are measurable continuous functions, $B(t)$ is Brownian motion defined on the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$. $\phi : [-\delta, 0] \to \mathbb{R}^n$ is a continuous function satisfying $\mathbb{E}|\phi(t)|^2 < \infty$. $x(t)$ denotes a n-dimensional random variable.

2. Preliminaries

For the sake of smooth follow-up work, we briefly give the preparatory work in this section.

Definition 2.1. (Definition 2.1, [1]) For any $\alpha \in (0, 1)$ and a function $f \in L^1[[0, T]; \mathbb{R}^n]$, the Riemann-Liouville fractional integral operator of order α is defined for all $0 < t < T$ by

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds,\ t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.1. *(Lemma 2.1, [1]) Let $f(t)$ be a continuous function, then its integration with respect to $(dt)^\alpha$, $0 < \alpha \leq 1$ is defined by*

$$\int_0^t f(s)(ds)^\alpha = \alpha \int_0^t (t-s)^{\alpha-1} f(s)ds,\ 1 \geq \alpha > 0.$$

Definition 2.2. An R^n-value stochastic process $x(t)_{-\tau\leq t\leq T}$ is called a unique solution to SFDEs (1.1) if $x(t)$ satisfies the following:
(i) $x(t)$ a continuous process of adaptation;

(ii) $b(t,x(t),y(t))$, $\sigma_1(t,x(t),y(t))\in\mathcal{L}^1(J;\mathbb{R}^n)$ and $\sigma_2(t,x(t),y(t))\in\mathcal{L}^2(J;\mathbb{R}^{n\times m})$;
(iii) For all $t\in[-\tau,T]$, $x(t)$ satisfies the following integral equation:

$$x(t)=\begin{cases}\phi_0+\int_0^t b\left(s,x(s),x(s-\delta(s))\right)ds+\int_0^t\sigma_1\left(s,x(s),x(s-\delta(s))\right)dB(s)\\ \qquad +\alpha\int_0^t(t-s)^{\alpha-1}\sigma_2\left(s,x(s),x(s-\delta(s))\right)ds,\ t\in J\\ \phi(t),t\in[-\delta,0],\end{cases}\tag{2.1}$$

where $x(0)=\phi_0=\{\phi(t),\ -\delta\le t\le 0\}$;
(iv) For all $t\in[-\delta,T]$, other solution $\hat{x}(t)$, we have $\mathbb{P}\{x(t)=\hat{x}(t)\}=1$.

Lemma 2.2. *(Lemma 2.4, [11]) Let $m\in\mathbb{N}$ and $x_1,\ x_2,\ \cdots,\ x_m$ be nonnegative real numbers. Then*

$$\left(\sum_{i=1}^m x_i\right)^p\le m^{p-1}\sum_{i=1}^m x_i^p,\ \ for\ p>1.$$

3. Existence and uniqueness of solutions to SFDEs

In this section, we will prove the existence and uniqueness of solutions to SFDEs (1.1).

In order to attain the solution of SFDEs (1.1), we impose the following two hypotheses.
(A1) (Non-Lipschitz condition). There exists a function $G(t,x,y):[0,+\infty)\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ such that:
(i) For any fixed $x,y\le 0$, $t\in[0,+\infty)\mapsto G(t,x,y)\in\mathbb{R}^+$ is locally integrable, and for any fixed $t\ge 0$, $x,y\in\mathbb{R}^+\mapsto G(t,x,y)\in\mathbb{R}^+$ is continuous, non-decreasing, concave, and satisfy $G(t,0,0)=0$ and for any fixed t, $\int_{0+}\int_{0+}\frac{1}{G(t,x,y)}dxdy=+\infty$.
(ii) For any fixed $t\ge 0$ and $x,y\in\mathbb{R}^n$, the following inequality holds:

$$\begin{aligned}&|b(t,x_1,y_1)-b(t,x_2,y_2)|^2+|\sigma_1(t,x_1,y_1)-\sigma_1(t,x_2,y_2)|^2\\&+|\sigma_2(t,x_1,y_1)-\sigma_2(t,x_2,y_2)|^2\le G\left(t,|x_1-y_1|^2,|x_2-y_2|^2\right).\end{aligned}$$

(iii) For every $t\in\mathbb{R}^+$ and any two non-negative functions $X(t)$, $Y(t)$ such that

$$AX(t)+Y(t)\le K\int_0^t G(s,X(s),Y(s))ds,$$

where A and K are non-negative constant, we have $X(t)$ and $Y(t)\equiv 0$.
(A2) Let $b(t,0,0)$, $\sigma_1(t,0,0)$, $\sigma_2(t,0,0)\in L^2([0,T])$ and for all $t\in[0,T]$, it follows that

$$|b(t,0,0)|^2+|\sigma_1(t,0,0)|^2+|\sigma_2(t,0,0)|^2\le K_b,$$

where $K_b\ge 0$ is a constant.

For any integer $n\ge 1$, define $x_n(t)=\phi(t)$ for all $-\tau\le t\le 0$ and

$$\begin{aligned}x_n(t)=&\phi_0+\int_0^t b\left(s,x_n\left(s-\frac{1}{n}\right),x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)ds\\&+\int_0^t\sigma_1\left(s,x_n\left(s-\frac{1}{n}\right),x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)dB(s)\\&+\alpha\int_0^t(t-s)^{\alpha-1}\sigma_2\left(s,x_n\left(s-\frac{1}{n}\right),x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)ds,\quad 0\le t\le T,\end{aligned}\tag{3.1}$$

where $\phi_0 = \phi(0)$.

Theorem 3.1. *Under the hypotheses A1-A2, there exists a unique solution $x(t)$ to SFDEs (1.1).*

Proof. The proof will be split into three steps.
Step 1. The boundedness of the sequence $\{x_n(t),\ n \geq 1\}$.
By Lemma 2.2, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t}|x_n(s)|^2\right)\\
=&4\mathbb{E}\left|\phi_0\right|^2 + 4\mathbb{E}\left|\int_0^t b\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)ds\right|^2\\
&+4\mathbb{E}\left(\sup_{0\leq s\leq t}\left|\int_0^s \sigma_1\left(u, x_n\left(u-\frac{1}{n}\right), x_n\left(u-\frac{1}{n}-\delta(u)\right)\right)dB(u)\right|^2\right)\\
&+4\alpha^2\mathbb{E}\left|\int_0^t (t-s)^{\alpha-1}\sigma_2\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)ds\right|^2.
\end{aligned}$$

By Lemma 2.2, Burkholder-Davis-Gundy inequality, Cauchy-Schwarz inequality and hypotheses A1-A2, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t}|x_n(s)|^2\right)\\
\leq&4\mathbb{E}\left|\phi_0\right|^2 + 8T\int_0^t \mathbb{E}\left(\left|b\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)\right.\right.\\
&\qquad\left.\left.-\,b(s,0,0)\right|^2 + |b(s,0,0)|^2\right)ds\\
&+32\int_0^t \mathbb{E}\left(\left|\sigma_1\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)\right.\right.\\
&\qquad\left.\left.-\,\sigma_1(s,0,0)\right|^2 + |\sigma_1(s,0,0)|^2\right)ds\\
&+8\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\left(\left|\sigma_2\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)\right.\right.\\
&\qquad\left.\left.-\,\sigma_2(s,0,0)\right|^2 + |\sigma_2(s,0,0)|^2\right)ds\\
\leq&4\mathbb{E}\left|\phi_0\right|^2 + 8\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\\
&\times\int_0^t\left(K_b + G\left(s, \mathbb{E}\left|x_n\left(s-\frac{1}{n}\right)\right|^2, \mathbb{E}\left|x_n\left(s-\frac{1}{n}-\delta(s)\right)\right|^2\right)\right)ds.
\end{aligned}$$

Given that $G(t,x,y)$ is concave, there exist $a_3(t)\geq 0$, $a_4(t)\geq 0$, $a_5(t)\geq 0$ such that

$$\begin{aligned}
&G(t,x,y) \leq a_3(t) + a_4(t)x + a_5(t)y,\ x,\ y\geq 0,\\
&\int_0^T a_3(t)dt < \infty,\ \int_0^T a_4(t)dt < \infty,\ \int_0^T a_5(t)dt < \infty.
\end{aligned} \tag{3.2}$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|x_n(s)|^2\right)\\
\le &C_1+C_2\left(\int_0^t \mathbb{E}\left(\sup_{0\le s_1\le s}|x_n(s_1)|^2\right)ds+\int_0^t \mathbb{E}\left(\sup_{0\le s_1\le s}|x_n(s_1-\delta(s))|^2\right)ds\right),
\end{aligned}
$$

where $C_1 = 4\mathbb{E}\left|\phi_0\right|^2 + 8T\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\left(K_b+\sup\limits_{0\le t\le T} a_3(t)\right)$,
$C_2 = 8\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\max\left\{\sup\limits_{0\le t\le T} a_4(t),\ \sup\limits_{0\le t\le T} a_5(t)\right\}$.

Let $\Omega(s)=\mathbb{E}\left(\sup\limits_{0\le s_1\le s}|x_n(s_1)|^2\right)$, and $\mathbb{E}\left(\sup\limits_{-\delta\le s_1\le 0}|x_n(s_1)|^2\right)=0$, then we have

$$
\Omega(s-\delta(s))=\mathbb{E}\left(\sup_{0\le s_1\le s}|x_n(s_1-\delta(s))|^2\right).
$$

Hence, we have

$$
\Omega(t)\le C_1+C_2\left(\int_0^t\Omega(s)ds+\int_0^t\Omega(s-\delta(s))ds\right).
$$

Let $\Upsilon(t)=\sup\limits_{\theta\in[-\delta,t]}\Omega(\theta)$, for all $t\in[0,T]$, then $\Omega(t)\le\Upsilon(t)$, and $\Omega(t-\delta(s))\le\Upsilon(t)$. Therefore, one can obtain

$$
\Omega(t)\le C_1+2C_2\left(\int_0^t\Upsilon(s)ds\right).
$$

Note that for all $\theta\in[0,t]$, we can obtain

$$
\Omega(\theta)\le C_1+2C_2\left(\int_0^\theta\Upsilon(s)ds\right)\le C_1+2C_2\left(\int_0^t\Upsilon(s)ds\right)
$$

and

$$
\Upsilon(t)=\sup_{\theta\in[-\delta,t]}\Omega(\theta)\le\max\left\{\sup_{\theta\in[-\delta,0]}\Omega(\theta),\ \sup_{\theta\in[0,t]}\Omega(\theta)\right\}\le C_1+2C_2\left(\int_0^t\Upsilon(s)ds\right).
$$

In terms of Gronwall-Bellman inequality, we get

$$
\Upsilon(t)\le C_1e^{2C_2t}.
$$

Then, we have

$$
\mathbb{E}\left(\sup_{0\le s\le t}|x_n(s)|^2\right)\le C_1e^{2C_2t}=C_3,
$$

where C_1, C_2 and C_3 are positive constants. Therefore, the boundedness of the sequence $\{x_n(t),\ n\ge 1\}$ has been proved.

Step 2. $\{x_n(t),\ n \geq 1\}$ is continuous on $[0,T]$. For $0 \leq s < t \leq T$ and any integer $n \geq 1$, by Lemma 2.2 and equation (3.1), we have

$$\begin{aligned}
&\mathbb{E}|x_n(t) - x_n(s)|^2 \\
\leq & 3\mathbb{E}\left|\int_s^t b\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right) du\right|^2 \\
&+ 3\mathbb{E}\left|\int_s^t \sigma_1\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right) dB(u)\right|^2 \\
&+ 3\alpha^2\mathbb{E}\Bigg|\int_s^t (t-u)^{\alpha-1}\sigma_2\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right) du \\
&+ \int_0^s \left[(t-u)^{\alpha-1} - (s-u)^{\alpha-1}\right]\sigma_2\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right) du\Bigg|^2.
\end{aligned}$$

With the help of Lemma 2.2, Itô isometry, Cauchy-Schwarz inequality and hypotheses A1-A2, we have

$$\begin{aligned}
&\mathbb{E}|x_n(t) - x_n(s)|^2 \\
\leq & 3(T-s)\int_s^t \mathbb{E}\left|b\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right)\right|^2 du \\
&+ 3\int_s^t \mathbb{E}\left|\sigma_1\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right)\right|^2 du \\
&+ 3\alpha^2\mathbb{E}\left(\sup_{0\leq u\leq t}\left|\sigma_2\left(u, x_n\left(u - \frac{1}{n}\right), x_n\left(u - \frac{1}{n} - \delta(s)\right)\right)\right|^2\right) \\
&\qquad \times \left|\int_s^t (t-u)^{\alpha-1} du + \int_0^s \left[(t-u)^{\alpha-1} - (s-u)^{\alpha-1}\right] du\right|^2 \\
\leq & 6\left(1 + T - s\right)\int_s^t \left(K_b + G\left(u, \mathbb{E}\left|x_n\left(u - \frac{1}{n}\right)\right|^2, \mathbb{E}\left|x_n\left(u - \frac{1}{n} - \delta(s)\right)\right|^2\right)\right) du \\
&+ 6\left[K_b + G\left(s, \mathbb{E}\left|x_n\left(u - \frac{1}{n}\right)\right|^2, \mathbb{E}\left|x_n\left(u - \frac{1}{n} - \delta(s)\right)\right|^2\right)\right]\left(t^\alpha - s^\alpha\right)^2.
\end{aligned}$$

From above inequality (3.2) and Step 1, we have

$$\mathbb{E}|x_n(t) - x_n(s)|^2 \leq C_4(t-s) + C_5\left(t-s\right)^{2\alpha}$$

where $C_5 = 6\left[K_b + \left(\sup_{0\leq t\leq T} a_3(t)\right) + 2C_3 \max\left\{\sup_{0\leq t\leq T} a_4(t),\ \sup_{0\leq t\leq T} a_5(t)\right\}\right]$, and $C_4 = C_5(1 + T - s)$.

Step 3. The sequence $\{X_n(t),\ n \geq 1\}$ is Cauchy sequence. For integer $1 \leq n < m$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq s\leq t}|x_n(s) - x_m(s)|^2\right) \\
\leq & 3\mathbb{E}\Bigg|\int_0^t b\left(s, x_n\left(s - \frac{1}{n}\right), x_n\left(s - \frac{1}{n} - \delta(s)\right)\right)
\end{aligned}$$

$$
\begin{aligned}
& - b\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right) ds\Big|^2 \\
&+ 3\mathbb{E}\Big|\int_0^t \sigma_1\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \\
&\qquad - \sigma_1\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right) dB(s)\Big|^2 \\
&+ 3\alpha^2\mathbb{E}\Big|\int_0^t (t-s)^{\alpha-1}\Big[\sigma_2\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \\
&\qquad - \sigma_2\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Big] ds\Big|^2.
\end{aligned}
$$

With the help of Lemma 2.2, Cauchy-Schwarz inequality and Burkholder-Davis-Gundy inequality, we have

$$
\mathbb{E}\left(\sup_{0\le s\le t}|x_n(s)-x_m(s)|^2\right) \tag{3.3}
$$

$$
\le 3T\int_0^t \mathbb{E}\Big|b\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \tag{3.4}
$$

$$
- b\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Big|^2 ds \tag{3.5}
$$

$$
+ 12\int_0^t \mathbb{E}\Big|\sigma_1\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \tag{3.6}
$$

$$
- \sigma_1\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Big|^2 ds \tag{3.7}
$$

$$
+ 3\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Big|\sigma_2\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \tag{3.8}
$$

$$
- \sigma_2\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Big|^2 ds. \tag{3.9}
$$

Applying the plus and minus technique as well as hypothesis A1(b) implies

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|x_n(s)-x_m(s)|^2\right) \\
\le& 6T\int_0^t \mathbb{E}\Big|b\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right) \\
&\qquad - b\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right)\Big|^2 ds \\
&+ 6T\int_0^t \mathbb{E}\Big|b\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right) \\
&\qquad - b\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Big|^2 ds \\
&+ 24\int_0^t \mathbb{E}\Big|\sigma_1\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad - \sigma_1\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right)\Bigg|^2 ds\\
&+ 24\int_0^t \mathbb{E}\Bigg|\sigma_1\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right)\\
&\qquad\qquad - \sigma_1\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Bigg|^2 ds\\
&+ 6\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Bigg|\sigma_2\left(s, x_n\left(s-\frac{1}{n}\right), x_n\left(s-\frac{1}{n}-\delta(s)\right)\right)\\
&\qquad\qquad - \sigma_2\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right)\Bigg|^2 ds\\
&+ 6\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\int_0^t \mathbb{E}\Bigg|\sigma_2\left(s, x_m\left(s-\frac{1}{n}\right), x_m\left(s-\frac{1}{n}-\delta(s)\right)\right)\\
&\qquad\qquad - \sigma_2\left(s, x_m\left(s-\frac{1}{m}\right), x_m\left(s-\frac{1}{m}-\delta(s)\right)\right)\Bigg|^2 ds\\
\le & 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, \mathbb{E}\Bigg(\sup_{0\le u\le s}\left|x_n\left(s-\frac{1}{n}\right)-x_m\left(s-\frac{1}{n}\right)\right|^2\Bigg),\\
&\qquad\qquad \mathbb{E}\Bigg(\sup_{0\le u\le s}\left|x_n\left(s-\frac{1}{n}-\delta(s)\right)-x_m\left(s-\frac{1}{n}-\delta(s)\right)\right|^2\Bigg)\Bigg)ds\\
&+ 6\left(4+5T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, \mathbb{E}\Bigg(\sup_{0\le u\le s}\left|x_m\left(s-\frac{1}{m}\right)-x_m\left(s-\frac{1}{n}\right)\right|^2\Bigg),\\
&\qquad\qquad \mathbb{E}\Bigg(\sup_{0\le u\le s}\left|x_m\left(s-\frac{1}{m}-\delta(s)\right)-x_m\left(s-\frac{1}{n}-\delta(s)\right)\right|^2\Bigg)\Bigg)ds.
\end{aligned}
$$

From Step 2, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le s\le t}|x_n(s)-x_m(s)|^2\right)\\
\le & 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, \mathbb{E}\left(\sup_{0\le u\le s}|x_n(u)-x_m(u)|^2\right),\\
&\qquad \mathbb{E}\left(\sup_{0\le u\le s}|x_n(u-\delta(u))-x_m(u-\delta(u))|^2\right)\Bigg)ds\\
&+ 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, C_4\left(\frac{1}{n}-\frac{1}{m}\right)\\
&\qquad + C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}, C_4\left(\frac{1}{n}-\frac{1}{m}\right)+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}\Bigg)ds.
\end{aligned}
$$

Let $\Omega_1(s) = \mathbb{E}\left(\sup_{0\le u\le s}|x_n(u)-x_m(u)|^2\right)$, and $\mathbb{E}\left(\sup_{-\tau\le u\le 0}|x_n(u)-x_m(u)|^2\right) = 0$, then we have

$$
\Omega_1(s-\delta(s)) = \mathbb{E}\left(\sup_{0\le u\le s}|x_n(u-\delta(u))-x_m(u-\delta(u))|^2\right).
$$

Hence, we have

$$\begin{aligned}\Omega_1(t) \leq& 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\left(s,\Omega_1(s),\Omega_1(s-\delta(s))\right)ds\\&+6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, C_4\left(\frac{1}{n}-\frac{1}{m}\right)\\&\qquad+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}, C_4\left(\frac{1}{n}-\frac{1}{m}\right)+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}\Bigg)ds.\end{aligned}$$

Let $\Upsilon_1(t)=\sup_{\theta\in[-\tau,t]}\Omega_1(\theta)$, for all $t\in[0,T]$, then $\Omega_1(t)\leq\Upsilon_1(t)$, and $\Omega_1(t-\delta(t))\leq\Upsilon_1(t)$. Therefore, one can obtain

$$\begin{aligned}\Omega_1(t) \leq& 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\left(s,\Upsilon_1(s),\Upsilon_1(s)\right)ds\\&+6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, C_4\left(\frac{1}{n}-\frac{1}{m}\right)\\&+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}, C_4\left(\frac{1}{n}-\frac{1}{m}\right)+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}\Bigg)ds.\end{aligned}$$

Note that for all $\theta\in[0,t]$, we can obtain

$$\begin{aligned}\Omega_1(\theta) \leq& 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\left(s,\Upsilon_1(s),\Upsilon_1(s)\right)ds\\&+6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, C_4\left(\frac{1}{n}-\frac{1}{m}\right)\\&\qquad+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}, C_4\left(\frac{1}{n}-\frac{1}{m}\right)+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}\Bigg)ds,\end{aligned}$$

and

$$\begin{aligned}\Upsilon_1(t)=&\sup_{\theta\in[-\tau,t]}\Omega_1(\theta)\leq\max\left\{\sup_{\theta\in[-\tau,0]}\Omega_1(\theta),\ \sup_{\theta\in[0,t]}\Omega_1(\theta)\right\}\\ \leq& 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\left(s,\Upsilon_1(s),\Upsilon_1(s)\right)ds\\&+6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\Bigg(s, C_4\left(\frac{1}{n}-\frac{1}{m}\right)\\&+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}, C_4\left(\frac{1}{n}-\frac{1}{m}\right)+C_5\left(\frac{1}{n}-\frac{1}{m}\right)^{2\alpha}\Bigg)ds.\end{aligned}\tag{3.10}$$

Let

$$\Upsilon_1(t)=\limsup_{n,m\to\infty}\mathbb{E}\left(\sup_{0\leq s\leq t}|x_n(s)-x_m(s)|^2\right).\tag{3.11}$$

Then, from above equations (3.10)-(3.11) together with Fatou's Lemma, it yields

$$\Upsilon_1(t)\leq A\Upsilon_1(t)+\Upsilon_1(t)\leq 6\left(4+T+\alpha^2\frac{T^{2\alpha-1}}{2\alpha-1}\right)\int_0^t G\left(s,\Upsilon_1(s),\Upsilon_1(s)\right)ds.\tag{3.12}$$

Finally, from equation (3.12) and hypothesis A1(c), we obtain

$$\Upsilon_1(t) = \limsup_{n,m\to\infty} \mathbb{E}\left(\sup_{0\le s\le t} |x_n(s) - x_m(s)|^2\right) = 0,$$

indicating that $\{x_n(t),\ n \ge 1\}$ is a Cauchy sequence. The Borel-Cantelli lemma shows that, as $n \to \infty$, $x_n(t) \to x(t)$ uniformly for $t \in [0,T]$. Hence, by taking limits on both sides of equation (3.1), we obtain that $x(t)$, $t \in [0,T]$, is a solution to equation (1.1) with the property $\mathbb{E}\left(\sup_{0\le s\le t} |x(s)|^2\right) < \infty$ for all $t \in [0,T]$, and this completes the proof of the existence. Now, the uniqueness of solution can be obtained by the same procedure as Step 3. Hence, the proof of Theorem 3.1 is completed. □

Remark 3.1. It should be pointed out that the addressed existence is more general than the existence concept investigated in [1]. In fact, our existence criterion can be reduced to that in [1] when $G(t, x(t), x(t)) = G(t, x(t))$.

4. Conclusion

In this work, the objective is to study the existence and uniqueness of FSDEs with time delays by using the novel Caratheodory approximation and the weaker non-Lipschitz condition. Our future work will focus on exploring Ulam-Hyers stability of various types of fractional differential equations with weaker conditions, and the explored conditions can be applied to a wider range of differential equations.

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