Journal of Mathematical Research & Exposition
Vol.23, No.2, 253–260, May, 2003

Hsu-Riordan Array/Partial Monoid *

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Abstract: This paper gives a unified approach to Hsu's two classes of extended GSN pairs in the setting of Hsu-Riordan partial monoid which is a generalization of Shapiro's Riordan group, and moreover Hsu-Wang transfer theorem, Brown-Sprugnoli transfer formula and generalized Brown transfer lemma which display some transfer methods of different kinds of Hsu-Riordan arrays and identities respectively.

Key words: Riordan array/Hsu-Riordan array; partial monoid; generating functon; Stirling numbers; transfer formula; identity.

Classification: AMS(2000) 05A10, 05A09/CLC number: O157.1

Document code: A **Article ID:** 1000-341X(2003)02-0253-08

1. Hsu-Riordan partial monoid

In this paper, we assume that $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}$, so that all what follows are formalities.

Definition 1.1 *For four complex functions $p(n,t), q(n,t), d(t)$, and $h(t)$, and an infinite matrix $A=(A(n,k))$, if*

$$d(t)p(k,h(t))=\sum_{n=0}^{\infty}A(n,k)q(n,t) \tag{1.1}$$

holds formally for $k=0,1,2,\cdots$, then we call A a Hsu-Riordan array, or HR-array in short, denoted by $(p(n,t),q(n,t);d(t),h(t))$, or$(p,q;d,h)$for short, viz., $(p,q;d,h):=(A(n,k))$.

Remark The original idea of (1.1) can be found in Hsu's masterpiece "Theory and application of generalized Stirling number pairs". This is the reason why we refer to the above matrix $(A(n,k))=:(p,q;d,h)$ as Hsu-Riordan array.

*Received date: 1999-01-10; Revised date: 2003-02-13
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If $f(t) := \sum_{k=0}^{\infty} f_k p(k,t)$, since

$$\sum_{n=0}^{\infty}(\sum_{k=0}^{\infty} A(n,k) f_k) q(n,t) = \sum_{k=0}^{\infty} f_k d(t) p(k,h(t)) = d(t) f(h(t)) \quad (\text{by}(1.1)), \tag{1.2}$$

we obtain

$$\sum_{k=0}^{\infty} A(n,k) f_k = [q(n,t)] d(t) f(h(t)), \tag{1.3}$$

where $[q(n,t)]g(t)$ denotes g_n with $g(t) = \sum_n g_n q(n,t)$. We call (1.2) Bruno-type formula which is a pattern of identity.

If we regard $\sum_k f_k p(k,t)$ as the generating function of $\{f_k\}_k$ with respect to $\{p(k,t)\}_k$, then (1.2) tells us that the generating function of the product of A and column vector $(f_0, f_1, \cdots)^T$ with respect to $\{q(n,t)\}_n$ is $d(t)f(h(t))$. Denote again this generating funtion by $(p,q;d,h) * [f|p]$ in which $[f|p]$ denotes f with $f(t) = \sum_k f_k p(k,t)$. Then

$$(p,q;d,h) * [f|p] = [df(h)|q]. \tag{1.4}$$

For two HR-arrays $(p,q;d,h)$ and $(q,r;f,g)$, and any sequence $\{F_k\}_k$, since

$$(q,r;f,g)(p,q;d,h) * [F|p] = (q,r;f,g) * [dF(h)|q] = [fd(g)F(h(g))|r],$$

we have by (1.4) that $(q,r;f,g)(p,q;d,h) = (p,r;fd(g),h(g))$. If we define

$$(q,r)(p,q) := (p,r), \tag{1.5}$$

$$(f,g)(d,h) := (fd(g),h(g)), \tag{1.6}$$

then

$$(q,r;f,g)(p,q;d,h) = ((q,r)(p,q);(f,g)(d,h)). \tag{1.7}$$

Definition 1.2 *A partial monoid is a triple $(M,p,1)$ in which M is a non-vacuous set, p is an associative partial binary product in M (viz., we admit that $p(a,b)$ is meanless for some $a,b \in M$), and 1 is an element of M such that $p(1,a) = a = p(a,1)$ for all $a \in M$.*

Denote

$$H := \{(p,q;d,h) | (p,q;d,h) \text{ is a HR} - \text{array}\},$$

$$M := \{(p,q) | \exists (d,h) s.t. (p,q;d,h) \text{ is a HR} - \text{array}\},$$

$$G := \{(d,h) | \exists (p,q) s.t. (p,q;d,h) \text{ is a HR} - \text{array}\}.$$

Obviously, G is a monoid (for monoid, see [12]) with (1.6) and the identity element $(1,t)$, M is a partial monoid with (1.5) and the identity element (p,p) for all p (more precisely, the quotient set $M/\sim$ becomes a partial monoid with $\overline{(q,r)}\ \overline{(p,q)} = \overline{(p,r)}$ and the identity element $\overline{(p,p)}$, where $\overline{(p,p)} = \{(q,q)|$ for any $q\}, \overline{(p,q)} = \{(p,q)\}$, if $p \neq q$,) and H is a partial monoid with (1.7) and the identity element $(p,p;1,t)$, unit matrix.We refer to H as a Hsu-Riordan partial monoid, and regard roughly H as the direct product of M and G in the natural way, i.e.,

$$H = M \otimes G. \tag{1.8}$$

Denote

$$H_p := \{(p,p;d,h)|(p,p;d,h) \in H\},$$

$$H_p^g := \{(p,p;d,h)|(p,p;d,h) \in H \text{ and } d,h \text{ are } fps \text{ with } o(d)=0, o(h)=1\},$$

where "fps" represents" formal power series" and

$$o(f(t)) := \min\{k|(\frac{f(t)}{t^k})_{t=0} \neq 0, k=0,1,2,\cdots\}.$$

It is easy to understand that the following holds.

Theorem 1.3 *H_p is a monoid, and H_p^g is an isomorphic image of Riordan group*[14--16].

When $p(n,t)$ is a fps with $o(p(n,t)) = n$, H_p^g is just a relative Riordan group, and accordingly, the element of H_p^g is a relative Riordan array in [18].

Definition 1.4 *If two HR-arrays $(A(n,k)) = (p,q;d,h)$ and $(B(n,k)) = (p_1,q_1;d_1,h_1)$ are inverses of each other, then we call $\langle A(n,k), B(n,k)\rangle$ a Hsu-Stirling number pair, or HSN pair for short.*

For HSN pair $\langle A(n,k), B(n,k)\rangle$, if d and h are fps, then

$$(B(n,k)) = (q,p;d^{-1}(\bar{h}),\bar{h}),$$

where $\bar{h}$ represents the compositional inverse of h. If, furthermore, $\{p(n,t)\}_n$ and $\{q(n,t)\}_n$ are two normal basic sets of polynomials (see [7]), and $d(t) = 1$, then the HSN pair is just Hsu's first class of extended GSN pair in [7].

If $(p,p;d,h) \in H_p^g$, then HSN pair $\langle A(n,k), B(n,k)\rangle$ becomes a HRSN pair [18]; because Hsu's second class of extended GSN pair in [7] is also a HRSN pair, so Hsu's second class of extended GSN pair is a HSN pair.

Summarizing above discussion, we obtain the conclusion that the Hsu-Riordan partial monoid is a proper setting for unifying Hsu's two classes of extended GSN pairs and some related results such as [6–9], which illustrates the principle of the intrinsic unity of one's thoughts.

2. Hsu-Wang transfer theorem

For given two fps $d(t)$ and $h(t)$, define

$$d(t)h(t)^m =: \sum_{i=0}^{\infty} f_{i,m}t^i,$$

$$h(t)^m =: \sum_{i=0}^{\infty} g_{i,m}t^i,$$

$$\mu_{(\alpha)}(p^i) := \lambda^i f_{i,\alpha(p)}, \gamma_{(\beta)}(p^i) := \lambda^i g_{i,\beta(p)},$$

$$\mu_{(\alpha)}(n) := \prod_{(*)} \mu_{(\alpha)}(p^i), \gamma_{(\beta)}(n) := \prod_{(*)} \gamma_{(\beta)}(p^i),$$

where $(*)$ represents the $(p^i|n$ and $p^{i+1} \nmid n)$, p is a prime, α and β are arithmetic functions. Then we have the following theorem.

Theorem 2.1 $(\mu_{(\alpha)} * \gamma_{(\beta)})(n) = \mu_{(\alpha+\beta)}(n)$, *where* $\mu_{(\alpha)} * \gamma_{(\beta)}$ *represents the Dirichlet product (see [13]).*

Proof For any prime p, we have

$$\begin{aligned}(\mu_{(\alpha)} * \gamma_{(\beta)})(p^e) &= \sum_{d|p^e} \mu_{(\alpha)}(d)\gamma_{(\beta)}(\frac{p^e}{d}) = \sum_{i=0}^{e} \mu_{(\alpha)}(p^i)\gamma_{(\beta)}(p^{e-i}) \\ &= \sum_{i=0}^{e} \lambda^i f_{i,\alpha(p)}\lambda^{e-i} g_{e-i,\beta(p)} = \lambda^e \sum_{i=0}^{e} f_{i,\alpha(p)} g_{e-i,\beta(p)} \\ &= \lambda^e f_{e,(\alpha+\beta)(p)} = \mu_{(\alpha+\beta)}(p^e).\end{aligned}$$

It is easy to see from the definition of $\mu_{(\alpha)}$ and $\gamma_{(\beta)}$ that they are multiplicative functions. Therefore their Dirichlet product $\mu_{(\alpha)} * \gamma_{(\beta)}$ is also mutiplicative, and hence

$$(\mu_{(\alpha)} * \gamma_{(\beta)})(n) = \mu_{(\alpha+\beta)}(n)$$

for any natural number n. This completes the proof.

If we define

$$f(\alpha) := \sum_{n=1}^{\infty} \frac{\mu_{(\alpha)}(n)}{n^s}, \quad g(\beta) := \sum_{n=1}^{\infty} \frac{\gamma_{(\beta)}(n)}{n^s},$$

then Theorem 2.1 tells us that $f(\alpha)g(\beta) = f(\alpha+\beta)$, and so $f(n) = f(1)g(1)^{n-1}$ for $n \geq 1$. $(A(n,m)) = (\mu_{(m+1)}(n))$ is a HR-array with the row vector of the generating functions of the columns of $(A(n,m))$ with respect to $\{n^{-t}\}_n$

$$(f(1), f(1)g(1), f(1)g(1)^2, \cdots).$$

Hence, Theorem 2.1 implies a general method for constructing new HR-array from the ordinary HR-arrays $(f_{i,m})$ and $(g_{i,m})$.

When $d(t) = 1, h(t) = \sum_{i=0}^{\infty} f_i t^i$, we obtain

Corollary 2.2 *If define*

$$(\sum_{i=0}^{\infty} f_i t^i)^m =: \sum_{i=0}^{\infty} f_{i,m} t^i,$$

$$\mu_{(\alpha)}(p^i) := \lambda^i f_{i,\alpha(p)},$$

with $f_0 = f_{0,0} = 1, f_{i,0} = 0 (i \geq 1)$, *and moreover*

$$\mu_{(\alpha)}(n) := \prod_{(*)} \mu_{(\alpha)}(p^i),$$

and

$$\mu_{(0)}(n) := [\frac{1}{n}] = \delta_{n,1},$$

where α is an arithmetic function, $(*)$ represents ($p^i|n$ and $p^{i+1}\nmid n$), and $\delta_{n,1}$ denotes the Kronecker-delta, then $(\mu_{(\alpha)} * \mu_{(\beta)})(n) = \mu_{(\alpha+\beta)}(n)$.

Corollary 2.3 *$F(n) = \sum_{d|n} \mu_{(\alpha)}(\frac{n}{d})G(d) \Longleftrightarrow G(n) = \sum_{d|n} \mu_{(-\alpha)}(\frac{n}{d})F(d)$ with $\mu_{(\alpha)}(n)$ defined in Corollary 2.2.*

In particular, since $f_k = \frac{1}{1+kz}\binom{1+kz}{k}$ in Corollary 2.2 implies

$$f_{k,m} = \frac{m}{m+kz}\binom{m+kz}{k}$$

on the basis of

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\frac{c}{c+b(n-k)}\binom{c+b(n-k)}{n-k} = \frac{a+c}{a+c+bn}\binom{a+c+bn}{n} \quad \text{(see [5],[10])}$$

and

$$f(x)f(y) = f(x+y) \Rightarrow f(x) = f(1)^x,$$

we know by taking

$$f_k = \frac{1}{1+kz}\binom{1+kz}{k}$$

and $\lambda = -1$ in Corollary 2.2 and 2.3, that the two corollaries become Theorems 2 and 3 in [10], respectively.

We refer to Theorem 2.1 as a Hsu-Wang transfer theorem to show the source of the theorem.

3. Generalized Brown transfer lemma

Theorem 3.1 *For given fps $f(x)$ and $h(x)$ with $o(h(x)) = 0$, define*

$$f(x)h(x)^k =: \sum_{n=0}^{\infty} C_{n,k}\left(\frac{x}{h(x)}\right)^n.$$

Then for sequences $\{\Phi_n\}_n$ and $\{\Psi_n\}_n$,

$$\Psi_n = \sum_{k=0}^{n} C_{n-k,k}\Phi_k \Longrightarrow \sum_{n=0}^{\infty} \Psi_n\left(\frac{x}{h(x)}\right)^n = f(x)\sum_{n=0}^{\infty}\Phi_n x^n.$$

If we take $f(x) = (1+x)^\alpha, h(x) = (1+x)^\beta$, then

$$C_{n,k} = \frac{\alpha+k\beta}{\alpha+(k+n)\beta}\binom{\alpha+(k+n)\beta}{n}$$

by the well-known Lagrange inversion theorem (see [4]). For given sequence$\{\lambda_k\}_k$, let $\Phi_k = \frac{\alpha}{\alpha+\beta k}\lambda_k, r_n = \sum_{k=0}^{n}\binom{\alpha+\beta n}{n-k}\lambda_k$, then

$$\Psi_n = \sum_{k=0}^{n} C_{n-k,k}\Phi_k = \frac{\alpha}{\alpha+\beta n}r_n,$$

and, Theorem 3.1 becomes

Corollary 3.2

$$r_n = \sum_{k=0}^{n} \binom{\alpha+\beta n}{n-k} \lambda_k \Longrightarrow \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n} r_n [\frac{x}{(1+x)^{\beta}}]^n = (1+x)^{\alpha} \sum_{n=0}^{\infty} \frac{\alpha}{\alpha+\beta n} \lambda_n x^n.$$

This is just the (3) of the lemma in [1]. We call Theorem 3.1 the generalized Brown transfer lemma.

4. Brown-Sprugnoli transfer formula

If p, q, d, h are fps, then we call $(p, q; d, h)$ a fps-type HR-array. In this section, we consider only the case in which $o(p(n,t)) = o(q(n,t)) = n, o(d) = 0, o(h) = 1$, and denote $p(n,t)$ and $q(n,t)$ by $p_n(t)$ and $q_n(t)$ respectively.

Let

$$d(t)p_k(h(t)) = \sum_{n \geq k} A(n,k) q_n(t),$$

$$d(t)P_k(h(t)) = \sum_{n \geq k} B(n,k) Q_n(t),$$

and

$$(p(t)) = (P(t))N, \tag{4.1}$$

$$(q(t)) = (Q(t))M, \tag{4.2}$$

where $(p(t))$ denotes the row vector $(p_0(t), p_1(t), p_2(t), \cdots)$. Since

$$(d(t)p(h(t))) = (q(t))A, \tag{4.3}$$

$$(d(t)P(h(t))) = (Q(t))B, \tag{4.4}$$

we have

$$\begin{aligned} (d(t)p(h(t))) &= (d(t)P(h(t)))N \quad (\text{by } (4.1)) \\ &= (Q(t))BN \quad (\text{by } (4.4)); \end{aligned} \tag{4.5}$$

$$\begin{aligned} (d(t)p(h(t))) &= (q(t))A \quad (\text{by } (4.3)) \\ &= (Q(t))MA \quad (\text{by } (4.2)), \end{aligned} \tag{4.6}$$

and hence

$$BN = MA \quad (\text{by } (4.5) \text{ and } (4.6)), \tag{4.7}$$

or

$$MAN^{-1} = B. \tag{4.8}$$

Theorem 4.1 *For given two HR-arrays $A = (p, q; d, h)$ and $B = (P, Q; d, h)$, if $(p) = (P)N$ and $(q) = (Q)M$, then $MA = BN$ or $MAN^{-1} = B$.*

Theorem 4.1 tells us a method of transforming a fps-type HR-array to another one.

If we take $p = q$ and $P_n(t) = Q_n(t) = t^n$, then $M = N$ and Theorem 4.1 becomes

Corollary 4.2 *If* $(p(t)) = (t)M$, *then*

$$M(p,p;d,h)M^{-1} = (t,t;d,h).$$

In particular, if $p_k(t) = \frac{t^k}{c_k}, c_k \neq 0$ for $k = 0,1,2,\cdots$, then $M = \mathrm{diag}(c_0^{-1}, c_1^{-1}, c_2^{-1}, \cdots)$. If $(p,p;d,h) = (h_{ij})$, then

$$M(h_{ij})M^{-1} = (\frac{c_j}{c_i}h_{ij}) \tag{4.9}$$

is a Riordan array. This is an answer to Shapiro's problem in [14] of developing a Riordan array-type theory with respect to generating functions of the form $A(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{c_n}$ for various sequences c_n.

Specially, if we take $c_k = k!$, then $(\frac{j!}{i!}h_{ij})$ is a Riordan array, where, h_{ij} satisfies

$$d(t)\frac{h(t)^j}{j!} = \sum_{i=0}^{\infty} h_{ij}\frac{t^i}{i!}.$$

For example, $(S(n,k)) = (\frac{t^n}{n!}, \frac{t^n}{n!}; 1, e^t - 1)$, so

$$(\frac{k!}{n!}S(n,k)) = (t^n, t^n; 1, e^t - 1). \tag{4.10}$$

where $S(n,k)$ represents the Stirling numbers of the second kind (see [4]). Thus

$$\begin{aligned}\sum_{1\le k\le n} S(n,k)y^k &= \sum_{0\le k\le n} S(n,k)\frac{k!}{n!}\frac{n!}{k!}y^k = n![t^n]e^{y(e^t-1)}\\ &= e^{-y}[\frac{t^n}{n!}]e^{ye^t} = e^{-y}\sum_{r\ge 0}[\frac{t^n}{n!}]\frac{y^r e^{rt}}{r!}\\ &= e^{-y}\sum_{r\ge 1}\frac{r^n}{r!}y^r.\end{aligned}$$

This is just the result (4.2.16), a Dobinski-type formula, in [17].

In view of the fact that J.W.Brown gave (4.9) in the setting of the connection sequences (see [2,3]) and R.Sprugnoli used (4.10) in [15], we call Theorem 4.1 Brown-Sprugnoli transfer formula.

Acknowledgement The author wishes to thank professor L.C.Hsu for valuable conversations.

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Hsu-Riordan 阵 /partial monoid

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摘 要: 本文首先对 Shapiro 的 Riordan 群进行了推广，给出了 Hsu-Riordan partial monoid 的概念，然后在此框架内，对徐利治先生的两类扩展型广义 Stirling 数偶进行了统一处理；建立了 Hsu-Wang 转换定理， Brown-Sprugnoli 转换公式，以及广义 Brown 转换引理 — 它揭示了一些不同类型的 Hsu-Riordan 阵之间转换的方法，由此可产生大量的恒等式.

关键词: Riordan 阵 / 徐 -Riordan 阵；偏 monoid; 发生函数； stirling 数；转换公式；恒等式.