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A UNICITY THEOREM FOR ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES**

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Abstract

A unicity theorem concerning the total derivative for entire functions of several complex variables is proved.

Keywords Unicity theorem, Entire function, Total derivative
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§ 1. Introduction

Let f and g be two nonconstant entire functions on $\mathbf{C}^n$, $a \in \mathbf{C}$. If $f - a$ and $g - a$ have same zeros counting multiplicities, we denote it by $f = a \Leftrightarrow g = a$. In [8] H. X. Yi proved the following theorem.

Theorem A. *Let f and g be two nonconstant entire functions on the complex plane, and let k be a positive integer. If $f = 0 \Leftrightarrow g = 0$, $f^{(k)} = 1 \Leftrightarrow g^{(k)} = 1$, and $\delta(0, f) > 1/2$, then $f^{(k)} \cdot g^{(k)} \equiv 1$ unless $f \equiv g$.*

He also indicated that the assumption "$\delta(0, f) > 1/2$" is the best possible.

In this paper, we try to generalize this kind of theorem to the entire function of several complex variables. First we introduce the definition of total derivative.

Definition 1.1. *Let f be an entire function on $\mathbf{C}^n$, the total derivative Df of f is defined by*

$$Df(z) = \sum_{j=1}^{n} z_j f_{z_j}(z),$$

where $z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n$, f_{z_j} is the partial derivative of f with respect to z_j $(j = 1, 2, \cdots, n)$. The k-th order total derivative $D^k f$ of f is defined inductively by

$$D^k f = D(D^{k-1} f), \qquad k = 2, 3, \cdots .$$

In [2] and [3] we proved: If f is a transcendental entire function on $\mathbf{C}^n$, then for any positive integer k, $D^k f$ is also a transcendental entire function on $\mathbf{C}^n$. However the partial derivative may not have this property. The total derivative has also an interesting property that it does not change under the coordinate transformation (It is easy to be verified). The main result in this paper is the following

Theorem 1.1. *Let f and g be two nonconstant entire functions on $\mathbf{C}^n$, and let k be a positive integer. If $f = 0 \Leftrightarrow g = 0$, $D^k f = 1 \Leftrightarrow D^k g = 1$, and $\delta(0, f) > 1/2$, then $f \equiv g$.*

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§ 2 . Notations and Lemmas

For $z=(z_1,\cdots,z_n)\in\mathbf{C}^n$, define $|z|=(|z_1|^2+\cdots+|z_n|^2)^{1/2}$. Let

$$S_n(r)=\{z\in\mathbf{C}^n;\ |z|=r\},\qquad \overline{B}_n(r)=\{z\in\mathbf{C}^n;\ |z|\le r\}.$$

Set $d=\partial+\bar\partial$ and $d^c=(\partial-\bar\partial)/4\pi i$. We define

$$\omega_n(z)=dd^c\log|z|^2,\quad \sigma_n(z)=d^c\log|z|^2\wedge\omega_n^{n-1}(z),\quad \nu_n(z)=dd^c|z|^2.$$

Then $\sigma_n(z)$ is a positive measure on $S_n(r)$ with the total measure one. Let $a\in\mathbf{P}^1$. If $f^{-1}(a)\neq\mathbf{C}^n$, we denote by Z_a^f the a-divisor of f, write $Z_a^f(r)=\overline{B}_n(r)\cap Z_a^f$ and define

$$n_f(r,a)=r^{2-2n}\int_{Z_a^f(r)}\nu_n^{n-1}(z).$$

Then the counting function $N_f(r,a)$ is defined by

$$N_f(r,a)=\int_0^r[n_f(t,a)-n_f(0,a)]\frac{dt}{t}+n_f(0,a)\log r,$$

where $n_f(0,a)$ is the Lelong number of Z_a^f at the origin. Then Jensen's formula gives that

$$N_f(r,0)-N_f(r,\infty)=\int_{S_n(r)}\log|f(z)|\sigma_n(z)+O(1).$$

We define the proximity function $m_f(r,a)$ by

$$m_f(r,a)=\begin{cases}\displaystyle\int_{S_n(r)}\log^+\frac{1}{|f(z)-a|}\sigma_n(z), & \text{if } \ a\neq\infty;\\ \displaystyle\int_{S_n(r)}\log^+|f(z)|\sigma_n(z), & \text{if } \ a=\infty.\end{cases}$$

We also define the characteristic function $T_f(r)$ by

$$T_f(r)=m_f(r,\infty)+N_f(r,\infty).$$

The first main theorem states that (cf. [4, Chapter 4, A5.1])

$$T_f(r)=m_f(r,a)+N_f(r,a)+O(1).$$

Define

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m_f(r,a)}{T_f(r)}=1-\limsup_{r\to\infty}\frac{N_f(r,a)}{T_f(r)}.$$

We say f to be transcendental if

$$\lim_{r\to\infty}\frac{T_f(r)}{\log r}=\infty.$$

It is well known that an entire function f is not transcendental if and only if it is a polynomial (cf. [5]).

In this paper, E is always viewed as a set with finite Lebesgue measure in $[0,\infty)$, although it may vary in each appearance.

Lemma 2.1. (cf. [2] and [3, Lemma 2.2]) *Let f be a transcendental entire function on $\mathbf{C}^n$. Then for any positive integer k, $D^k f$ is also a transcendental entire function on $\mathbf{C}^n$, and*

$$m_{D^k f/f}(r,\infty) = O(\log rT_f(r)), \qquad r\bar{\in}E.$$

Lemma 2.2. (cf. [2, Theorem 3.1]) *Let f be a transcendental entire function on $\mathbf{C}^n$. Then for any positive integer k,*

$$T_f(r) \leq N_f(r,0) + N_{D^k f}(r,1) - N_{D^{k+1} f}(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E.$$

Lemma 2.3. *Let f be a transcendental entire function on $\mathbf{C}^n$. Then for any positive integer k,*

$$N_{D^k f}(r,0) \leq T_{D^k f}(r) - T_f(r) + N_f(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E.$$

Proof. Since

$$\frac{1}{f} = \frac{1}{D^k f}\cdot\frac{D^k f}{f},$$

we have

$$m_f(r,0) \leq m_{D^k f}(r,0) + m_{D^k f/f}(r,\infty). \tag{2.1}$$

Therefore, from Lemma 2.1 and the first main theorem we have

$$\begin{aligned} T_f(r) - N_f(r,0) &= m_f(r,0) + O(1) \leq m_{D^k f}(r,0) + O(\log rT_f(r)) \\ &= T_{D^k f}(r) - N_{D^k f}(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E. \end{aligned}$$

Hence

$$N_{D^k f}(r,0) \leq T_{D^k f}(r) - T_f(r) + N_f(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E.$$

Lemma 2.4. *Let f be a transcendental entire function on $\mathbf{C}^n$. Then for any positive integer k,*

$$T_{D^k f}(r) \leq T_f(r) + O(\log rT_f(r)), \qquad r\bar{\in}E, \tag{2.2}$$

$$N_{D^k f}(r,0) \leq N_f(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E. \tag{2.3}$$

Proof. From Lemma 2.1 we have

$$T_{D^k f}(r) = m_{D^k f}(r,\infty) \leq m_f(r,\infty) + m_{D^k f/f}(r,\infty) = T_f(r) + O(\log rT_f(r)), \qquad r\bar{\in}E,$$

which deduces (2.2). Hence

$$T_{D^k f}(r) - T_f(r) \leq O(\log rT_f(r)), \qquad r\bar{\in}E.$$

From Lemma 2.3 and the above inequality we get (2.3).

Lemma 2.5. *Let f and g be two transcendental entire functions on $\mathbf{C}^n$. If $f = 0 \Leftrightarrow g = 0$ and $D^k f = 1 \Leftrightarrow D^k g = 1$, then*

$$T_g(r) = O(T_f(r)), \qquad r\bar{\in}E.$$

Proof. From Lemma 2.2 we have

$$T_g(r) \le N_g(r,0) + N_{D^k g}(r,1) + O(\log rT_g(r)), \qquad r \bar{\in} E.$$

Since $N_g(r,0) = N_f(r,0)$ and $N_{D^k g}(r,1) = N_{D^k f}(r,1)$, from the above inequality and (2.2) we have

$$\begin{aligned}(1+o(1))T_g(r) &\le N_g(r,0) + N_{D^k g}(r,1),\\ &= N_f(r,0) + N_{D^k f}(r,1) \le T_f(r) + T_{D^k f}(r) + O(1)\\ &\le 2T_f(r) + O(\log rT_f(r)) = (2+o(1))T_f(r), \qquad r \bar{\in} E.\end{aligned}$$

Hence

$$T_g(r) = O(T_f(r)), \qquad r \bar{\in} E.$$

Lemma 2.6. *Let f and g be two nonconstant entire functions on $\mathbf{C}^n$. If $f = 0 \Leftrightarrow g = 0$ and $D^k f = 1 \Leftrightarrow D^k g = 1$, then f is transcendental if and only if g is transcendental.*

Proof. Suppose that f is transcendental. If g were not transcendental, then it is a polynomial, hence $D^k g$ is also a polynomial. Therefore $T_g(r) = O(\log r)$ and $T_{D^k g}(r) = O(\log r)$.

Since $N_f(r,0) = N_g(r,0)$ and $N_{D^k f}(r,1) = N_{D^k g}(r,1)$, from Lemma 2.2 we have

$$\begin{aligned}T_f(r) &\le N_f(r,0) + N_{D^k f}(r,1) + O(\log rT_f(r))\\ &= N_g(r,0) + N_{D^k g}(r,1) + O(\log rT_f(r))\\ &\le T_g(r) + T_{D^k g}(r) + O(\log rT_f(r)) = O(\log rT_f(r)), \qquad r \bar{\in} E,\end{aligned}$$

which gives a contradiction. Hence g is transcendental.

In the same way we can prove that if g is transcendental, then f is transcendental.

Lemma 2.7. *Let f_1, f_2, f_3 be linearly independent entire functions on $\mathbf{C}^n$. If $f_1 + f_2 + f_3 \equiv 1$, then*

$$T(r) \le \sum_{j=1}^{3} N_{f_j}(r,0) + O(\log rT(r)), \qquad r \bar{\in} E,$$

where $T(r) = \max_{1 \le j \le 3} T_{f_j}(r)$.

Proof. Define a holomorphic map $f : \mathbf{C}^n \to P^2(\mathbf{C})$ by

$$f(z) = [f_1(z), f_2(z), f_3(z)].$$

As usual, we define the characteristic function of f by

$$T(r,f) = \int_{S_n(r)} \log \|f(z)\| \sigma_n(z) + \log \|f(0)\|.$$

Let $H = \{[z_1, z_2, z_3] \in P^2(\mathbf{C}) \mid a_1 z_1 + a_2 z_2 + a_3 z_3 = 0\}$ be a hyperplane in $P^2(\mathbf{C})$. We denote by $N_f(r,H)$ the counting function of the divisor defined by $a_1 z_1 + a_2 z_2 + a_3 z_3 = 0$.

Set

$$\begin{aligned}H_j &= \{[z_1, z_2, z_3] \in P^2(\mathbf{C}) \mid z_j = 0\}, \qquad j = 1,2,3,\\ H_4 &= \{[z_1, z_2, z_3] \in P^2(\mathbf{C}) \mid z_1 + z_2 + z_3 = 0\}.\end{aligned}$$

Since f_1, f_2, f_3 are linear independent, then map f is non-degenerate. Obviously, H_j $(j = 1, 2, 3, 4)$ are in general position, hence by the second main theorem (cf. [7, Theorem 2]) we have

$$T(r, f) \leq \sum_{j=1}^{4} N_f(r, H_j) + O(\log rT(r, f)). \tag{2.4}$$

Since $f_1 + f_2 + f_3 = 1$ and $\|f\| \geq |f_j|$ $(j = 1, 2, 3)$, we have $|f_1| + |f_2| + |f_3| \geq 1$, and $3\|f\| \geq |f_1| + |f_2| + |f_3| \geq 1$. Therefore

$$\begin{aligned} T_{f_j}(r) = m_{f_j}(r, \infty) &= \int_{S_n(r)} \log^+ |f_j(z)|\sigma_n(z) \\ &\leq \int_{S_n(r)} \log 3\|f(z)\|\sigma_n(z) \\ &= \int_{S_n(r)} \log \|f(z)\|\sigma_n(z) + \log 3 = T(r, f) + O(1), \qquad j = 1, 2, 3. \end{aligned}$$

Then we deduce that

$$T(r) \leq T(r, f) + O(1), \tag{2.5}$$

where $T(r) = \max_{1 \leq j \leq 3} T_{f_j}(r)$.

By the definition of characteristic function, it is easy to see that

$$T(r, f) \leq O(T(r)). \tag{2.6}$$

Since $f_1 + f_2 + f_3 = 1$, we have $N_f(r, H_4) = 0$. Obviously, $N_f(r, H_j) = N_{f_j}(r, 0)$ $(j = 1, 2, 3)$. Hence from (2.4), (2.5) and (2.6) we deduce the conclusion.

Lemma 2.8. *Let f_1, f_2, f_3 be three entire functions on $\mathbf{C}^n$, and let at least one of f_j $(j = 1, 2, 3)$ be transcendental. If $f_1 + f_2 + f_3 \equiv 1$, and*

$$\sum_{j=1}^{3} N_{f_j}(r, 0) \leq (\lambda + o(1))T(r), \qquad r \bar{\in} E,$$

where $T(r) = \max_{1 \leq j \leq 3} T_{f_j}(r)$ and the constant $\lambda < 1$, then f_1, f_2, f_3 are linearly dependent.

Proof. Since at least one of f_j $(j = 1, 2, 3)$ is transcendental, we have

$$\lim_{r \to \infty} \frac{T(r)}{\log r} = \infty.$$

Assume that f_1, f_2, f_3 were linearly independent. Then from Lemma 2.7 and the assumption we have

$$T(r) \leq \sum_{j=1}^{3} N_{f_j}(r, 0) + O(\log rT(r)) \leq (\lambda + o(1))T(r), \qquad r \bar{\in} E,$$

which gives a contradiction.

Lemma 2.9. *Let f and g be two entire functions on $\mathbf{C}^n$, and let k be a positive integer.*
(1) *If $D^k f$ is constant, then f is constant and $D^k f \equiv 0$;*
(2) *If $D^k f \equiv D^k g$, then $f \equiv g + c$, where c is a constant.*

Proof. (1) Since f is an entire function on $\mathbf{C}^n$, we have a convergent series on $\mathbf{C}^n$ as follows:

$$f(z)=\sum_{m=0}^{\infty}P^m(z),$$

where $P^m(z)$ is either identically zero or a homogeneous polynomial of degree m in z ($m=0,1,2,\cdots$). By the homogeneity of $P^m(z)$ we have

$$\sum_{j=1}^{n}z_jP^m_{z_j}(z)=mP^m(z),\qquad m=1,2,\cdots.$$

Hence we see that

$$Df(z)=\sum_{j=1}^{n}z_jf_{z_j}(z)=\sum_{m=1}^{\infty}mP^m(z).$$

By induction, we have

$$D^kf(z)=\sum_{m=1}^{\infty}m^kP^m(z).$$

If D^kf is constant, every $m^kP^m(z)$ must be identically zero, so is $P^m(z)$ ($m=1,2,\cdots$). Thus f is constant and $D^kf\equiv 0$.

(2) In the same way as (1), we have

$$g(z)=\sum_{m=0}^{\infty}\widetilde{P}^m(z),$$

where $\widetilde{P}^m(z)$ is either identically zero or a homogeneous polynomial of degree m in z ($m=0,1,2,\cdots$), and

$$D^kg(z)=\sum_{m=1}^{\infty}m^k\widetilde{P}^m(z).$$

Since $D^kf\equiv D^kg$, we have

$$\sum_{m=1}^{\infty}m^k(P^m(z)-\widetilde{P}^m(z))\equiv 0.$$

Since $P^m(z)-\widetilde{P}^m(z)$ is either identically zero or a homogeneous polynomial of degree m in z ($m=1,2,\cdots$), then $P^m(z)-\widetilde{P}^m(z)\equiv 0$ ($m=1,2,\cdots$). Therefore

$$f\equiv g+c.$$

Lemma 2.10. *Let f_1,f_2 be two nonconstant entire functions on $\mathbf{C}^n$, and let c_1,c_2,c_3 be three nonzero constants. If $c_1f_1+c_2f_2=c_3$, then*

$$T(r)\le N_{f_1}(r,0)+N_{f_2}(r,0)+O(\log rT(r)),\qquad r\bar{\in}E,$$

where $T(r)=\max\{T_{f_1}(r),T_{f_2}(r)\}$.

Proof. By the second main theorem for the holomorphic functions, we have

$$T_{f_1}(r)\le N_{f_1}(r,0)+N_{f_1}(r,c_3/c_1)+O(\log rT_{f_1}(r)),\qquad r\bar{\in}E.$$

Noticing that $N_{f_1}(r, c_3/c_1) = N_{f_2}(r, 0)$, we have

$$T_{f_1}(r) \le N_{f_1}(r,0) + N_{f_2}(r,0) + O(\log rT(r)), \qquad r\bar{\in}E.$$

Similarly, we have

$$T_{f_2}(r) \le N_{f_1}(r,0) + N_{f_2}(r,0) + O(\log rT(r)), \qquad r\bar{\in}E.$$

Hence we get the conclusion.

§ 3 . Proof of Theorem 1.1

First we consider the polynomial case.

Lemma 3.1. *Let f and g be two nonconstant entire functions on $\mathbf{C}^n$, and let k be a positive integer. If $f = 0 \Leftrightarrow g = 0$, $D^k f = 1 \Leftrightarrow D^k g = 1$, and f is a polynomial, then $f \equiv g$.*

Proof. Since f is a polynomial, from Lemma 2.6 g is also a polynomial. Set

$$h = \frac{D^k f - 1}{D^k g - 1},$$

hence

$$D^k f - 1 = h(D^k g - 1).$$

Then h is a nowhere zero entire function. Since $D^k f - 1$ and $D^k g - 1$ are polynomials, we have that h is a nowhere zero polynomial, hence h is a constant.

Notice that $D^k f(0) = D^k g(0) = 0$, hence $h \equiv 1$, therefore $D^k f \equiv D^k g$. From Lemma 2.9 we have $f \equiv g + c$. Notice that f and g are nonconstant polynomials, from $f = 0 \Leftrightarrow g = 0$ we deduce $f \equiv g$.

Lemma 3.2. *Assume that the conditions of Theorem* 1.1 *are satisfied, and f is a transcendental entire function on $\mathbf{C}^n$. Then*

$$T_f(r) = O(T_{D^k f}(r)), \qquad r\bar{\in}E.$$

Proof. By the first main theorem, Lemma 2.1 and (2.2) we have

$$m_f(r,0) \le m_{D^k f}(r,0) + O(\log rT_f(r)) \le T_{D^k f}(r) + o(T_f(r)), \qquad r\bar{\in}E.$$

Since $\delta(a,f) = \liminf_{r\to\infty} \frac{m_f(r,a)}{T_f(r)} > 1/2$, when r is large enough we have $m_f(r,a) \ge \frac{1}{2}T_f(r)$. Hence from above inequality we have

$$\frac{1}{2}T_f(r) \le T_{D^k f}(r) + o(T_f(r)), \qquad r\bar{\in}E.$$

Thus

$$T_f(r) = O(T_{D^k f}(r)), \qquad r\bar{\in}E.$$

Proof of Theorem 1.1. From Lemma 3.1 we need only to prove the case when f is transcendental.

Let f be a transcendental entire function. Then from the assumption and Lemma 2.6, g is also a transcendental entire function. Set

$$h = \frac{D^k f - 1}{D^k g - 1}, \tag{3.1}$$

hence

$$D^k f - 1 = h(D^k g - 1).$$

Then h is a nowhere zero entire function. Let $f_1 = D^k f, f_2 = h, f_3 = -hD^k g$. Then

$$f_1 + f_2 + f_3 = 1, \tag{3.2}$$

and

$$\sum_{j=1}^{3} N_{f_j}(r,0) = N_{D^k f}(r,0) + N_{D^k g}(r,0). \tag{3.3}$$

From Lemma 2.3 we have

$$N_{D^k f}(r,0) \leq T_{D^k f}(r) - T_f(r) + N_f(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E. \tag{3.4}$$

And from Lemma 2.4 and Lemma 2.5 we have

$$N_{D^k g}(r,0) \leq N_g(r,0) + O(\log rT_g(r)) \leq N_g(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E. \tag{3.5}$$

Noticing that $N_g(r,0) = N_f(r,0)$, from (3.3)–(3.5) we have

$$\sum_{j=1}^{3} N_{f_j}(r,0) \leq T_{D^k f}(r) - T_f(r) + 2N_f(r,0) + O(\log rT_f(r)), \qquad r\bar{\in}E. \tag{3.6}$$

Since $\delta(0,f) = 1 - \limsup\limits_{r\to\infty} \frac{N_f(r,0)}{T_f(r)}$, we have $N_f(r,0) \leq (1-\delta(0,f)+o(1))T_f(r)$, so that from (3.6), (2.2) and Lemma 3.2 we deduce that

$$\begin{aligned}
\sum_{j=1}^{3} N_{f_j}(r,0) &\leq T_{D^k f}(r) - T_f(r) + [2(1-\delta(0,f))]T_f(r) + o(T_f(r)) \\
&\leq T_{D^k f}(r) - (2\delta(0,f)-1)T_f(r) + o(T_f(r)) \\
&\leq T_{D^k f}(r) - (2\delta(0,f)-1)T_{D^k f}(r) + o(T_{D^k f}(r)) \\
&= [2(1-\delta(0,f)) + o(1)]T_{D^k f}(r), \qquad r\bar{\in}E.
\end{aligned}$$

Since $f_1 = D^k f$, from the above inequality we can derive

$$\sum_{j=1}^{3} N_{f_j}(r,0) \leq (\lambda + o(1))T(r), \tag{3.7}$$

where $T(r) = \max\limits_{1\leq j\leq 3} T_{f_j}(r)$, and $\lambda = 2(1-\delta(0,f)) < 1$.

Hence from Lemma 2.8 we know that f_1, f_2, f_3 are linearly dependent, so there exist three constants, not all zero, such that

$$c_1 f_1 + c_2 f_2 + c_3 f_3 = 0. \tag{3.8}$$

From Lemma 2.1 we know that $f_1 = D^k f$ is transcendental. From Lemma 2.1 and Lemma 2.6, we can see that $D^k g$ is transcendental. Now we prove that at least one of f_2 and f_3 is constant.

Assume that f_2 and f_3 were not constants. First we prove that $c_1 \neq 0$ and $c_3 \neq 0$ under this assumption.

If $c_1 = 0$, then $c_2 f_2 + c_3 f_3 = 0$. If $c_2 = 0$, then $f_3 = -hD^k g \equiv 0$, so that $D^k g \equiv 0$, which contradicts the fact that $D^k g$ is transcendental. If $c_3 = 0$, then $f_2 = h \equiv 0$, which contradicts the assumption. Therefore, $c_2 \neq 0, c_3 \neq 0$. In this case

$$f_3 = -\frac{c_2}{c_3} f_2,$$

that is,

$$D^k g = \frac{c_2}{c_3} \neq 0.$$

However, $D^k g(0) = 0$, we get a contradiction. Hence $c_1 \neq 0$.

If $c_3 = 0$, then $c_1 f_1 + c_2 f_2 = 0$. It is easy to see that $c_1 \neq 0, c_2 \neq 0$. Hence

$$f_1 = -\frac{c_2}{c_1} f_2,$$

that is,

$$D^k f = -\frac{c_2}{c_1} h \neq 0.$$

However, $D^k f(0) = 0$, we get a contradiction. Hence $c_3 \neq 0$.

From (3.2) and (3.8) we have

$$\Big(1 - \frac{c_2}{c_1}\Big) f_2 + \Big(1 - \frac{c_3}{c_1}\Big) f_3 = 1.$$

Obviously, $c_2 = c_3 = c_1$ does not hold. If either $c_2 = c_1$ or $c_3 = c_1$, we can easily derive that f_2 or f_3 is constant, which contradicts the assumption. Hence $c_2 \neq c_1$ and $c_3 \neq c_1$. From Lemma 2.10 we have

$$T_{f_j}(r) \leq N_{f_2}(r,0) + N_{f_3}(r,0) + O(\log rT(r)), \qquad r\bar{\in} E, \quad j = 2,3, \tag{3.9}$$

where $T(r) = \max\limits_{1\leq j\leq 3} T_{f_j}(r)$.

From (3.2) and (3.8) we have

$$\Big(1 - \frac{c_1}{c_3}\Big) f_1 + \Big(1 - \frac{c_1}{c_3}\Big) f_2 = 1.$$

Obviously, $c_1 = c_2 = c_3$ does not hold. If either $c_1 = c_3$ or $c_2 = c_3$, we can easily derive that f_1 or f_2 is constant, which contradicts the assumption. Hence $c_2 \neq c_1$ and $c_3 \neq c_1$. From Lemma 2.10 we have

$$T_{f_j}(r) \leq N_{f_1}(r,0) + N_{f_2}(r,0) + O(\log rT(r)), \qquad r\bar{\in} E, \quad j = 1,2. \tag{3.10}$$

From (3.9) and (3.10) and (3.7) we have

$$T(r) \leq \sum_{j=1}^{3} N_{f_j}(r,0) + O(\log rT(r)) \leq (\lambda + o(1))T(r), \qquad r\bar{\in} E, \tag{3.11}$$

which is a contradiction.

Hence at least one of f_2 and f_3 is constant. If $f_3 = -hD^k g$ is a constant, from $D^k g(0) = 0$, we have $f_3 \equiv 0$. Since $h \neq 0$, $D^k g \equiv 0$. From Lemma 2.9, g is a constant, which contradicts the fact that g is transcendental.

Hence $f_2 = h$ is a constant. Since $D^k f - 1 = h(D^k g - 1)$ and $D^k f(0) = D^k g(0) = 0$, we have $h \equiv 1$, that is, $D^k f \equiv D^k g$. From Lemma 2.9 we deduce $f \equiv g + c$.

If $c \neq 0$, by the second main theorem we have

$$\begin{aligned} T_f(r) &\leq N_f(r,0) + N_f(r,c) + O(\log rT(r)) \\ &= N_f(r,0) + N_g(r,0) + O(\log rT(r)) \\ &= 2N_f(r,0) + O(\log rT(r)) \\ &\leq 2(1-\delta(0,f))T_f(r) + o(T_f(r)), \qquad r\bar{\in}E. \end{aligned}$$

Hence $2(1-\delta(0,f)) \geq 1$, which contradicts $\delta(0,f)) > 1/2$.

Therefore $c = 0$, that is, $f \equiv g$. The proof is completed.

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