Vol. 35 (2015)
No. 6

数 学 杂 志
J. of Math. (PRC)

ON TL-FUZZY IDEALS IN LATTICES

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Abstract: In this paper, we study TL-fuzzy ideals in lattices. By a TL-fuzzy ideal generated by an L-fuzzy subset, we prove that the lattice of T_M-fuzzy ideals in a modular lattice is a complete modular lattice. Moreover, using the projection and the cut shadow of an L-fuzzy set, we obtain necessary and sufficient conditions for a TL-fuzzy ideal of a Cartesian product of lattices to be a T-product of TL-fuzzy ideals of lattices. Our results generalize and develop the fuzzy ideal theory in lattices.

Keywords: left continuous t-norm; TL-fuzzy ideal; modular lattice; T-product

2010 MR Subject Classification: 06D72; 08A72

Document code: A **Article ID:** 0255-7797(2015)06-1341-12

1 Introduction

An important notion in fuzzy set theory is that of triangular norms: t-norms are used to define a generalized intersection. By a fuzzy subset μ in a given universe X we understand a mapping μ: $X \to [0,1]$, the membership degrees $\mu(x)$ can in a natural way be understood as the truth value (in fuzzy logic) of the statement "x belongs to μ". In the same way, the intersection $\mu \cap \nu$ of fuzzy sets μ, ν can be viewed as having the membership degree $(\mu\cap\nu)(x)$ corresponding to the truth degree of the statement "x belongs to μ" and "x belongs to ν". Here AND refers to a suitably defined conjunction connective, defined according to the different possibilities which one has to determine the membership degrees $(\mu \cap \nu)(x)$. For example, AND can be understood as taking the minimum or as taking the (usual, i.e., algebraic) product, or more generally, it also can be understood as a t-norm. Accordingly, the t-norms were considered as the candidates for generalized conjunction connectives of the background many-valued logic. The fuzzy logic based on t-norms, especially left continuous t-norms, was developed significantly by Hájek, Esteva et al. (see [1–4]). Also, there are many applications of triangular norms in several fields of mathematics and artificial intelligence [5].

In 1971, Rosenfeld [6] used the concept of fuzzy sets to formulate the notion of fuzzy groups. Since then, many other fuzzy algebraic concepts of fuzzy groups have been developed. Anthony and Sherwood [7] redefined fuzzy subgroups, which we call T-fuzzy subgroups

* **Received date:** 2013-04-24 **Accepted date:** 2013-09-23

Foundation item: Supported by Graduate Independent Innovation Foundation of Northwest University (YZZ12061); Scientific Research Program Funded by Shaanxi Provincial Education Department (2013JK0562).

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in this note, in terms of t-norm T which replaced the minimum operation and they [7, 8] characterized basic properties of T-fuzzy subgroups. In [9], Hu studied T-fuzzy groups with thresholds. Chon [10] characterized necessary and sufficient conditions whereby a fuzzy subgroup of a Cartesian product of groups is the product of fuzzy subgroups under minimum operation. He pointed out that finding necessary and sufficient conditions for T-fuzzy subgroups under a t-norm is still an open problem. In 2011, Yamak et al.[11] solved this open problem and identified necessary and sufficient conditions for TL-subgroups of a Cartesian product of groups, which can be represented as a T-product of TL-subgroups under t-norm operation. In the same paper, they also pointed out that the same problem could be studied in other algebraic structures such as rings and lattices.

In [6], the idea of a least fuzzy subgroupoid containing a given fuzzy set was also introduced. Consequently, Rosenfeld constructed the lattice of all fuzzy subgroupoids of a given group. In a recent paper [12], Jahan established that the lattice of all fuzzy ideals of a ring is modular. In fact, the proof of modularity is heavily based on the property of the unit interval that it is a dense chain. However, modularity of the lattices of L-normal subgroups of a group and L-ideals of a ring remains an open question. In 2011, Jahan [13] answered the question of modularity of the lattice of L-ideals of a ring.

With the development of theories of fuzzy algebra, Swamy [14] discussed the correspondence relation between fuzzy ideals and fuzzy congruences in a distributive lattice. In 2008, Koguep et al.[15] studied the notion of fuzzy prime ideal and highlighted the difference between fuzzy prime ideal and prime fuzzy ideal of a lattice. However, not much attention was paid to the studies of the lattices of fuzzy ideals of a lattice and the modularity of them.

The present work has been started as a continuation of these studies. In this paper, we will discuss modularity of the lattices of fuzzy ideals in a lattice. Moreover, we will explore necessary and sufficient conditions for a fuzzy ideal of a Cartesian product of lattices to be a T-product of fuzzy ideals of lattices under a left continuous t-norm T on a complete lattice L.

2 Preliminaries

In this section, we recall some notions and definitions that will be used in the sequel.

Let $(L, \wedge, \vee, \leq, 0, 1)$ denote a complete lattice with the top and bottom elements 1 and 0, respectively.

Definition 2.1 [5] A binary operation T on L is called a t-norm if it satisfies the following conditions: for any $a, b, c \in L$,

(T1) $aT1 = a$;

(T2) $aTb = bTa$;

(T3) $(aTb)Tc = aT(bTc)$;

(T4) if $b \leq c$, then $aTb \leq aTc$.

Because of associative and commutative properties, for any $a_1, a_2, \cdots, a_n \in L$ $(n \geq 1)$, $a_1Ta_2T\cdots Ta_n$ is well defined and its value is irrelevant to the order of $a_1, a_2, \cdots, a_n$. We write $T_{i=1}^{n} a_i = a_1 T a_2 T \cdots T a_n$. If $aT(\vee_{i\in I} b_i) = \vee_{i\in I}(aTb_i)$, for $a, b_i \in L$, where I is the set of natural numbers, then T is called a left continuous t-norm, see [2].

In what follows, let $L = [0, 1]$, there are some examples of the most popular t-norms: for any $x, y \in [0, 1]$,

(1) the Lukasiewicz t-norm: $xT_L y = \max\{x + y - 1, 0\}$;

(2) the algebraic product: $xT_P y = xy$;

(3) the standard min operation: $xT_M y = \min\{x, y\}$.

Throughout this paper, unless otherwise stated, $(L, \wedge, \vee, \leq, 0, 1)$ always represents a given complete lattice with a left continuous t-norm T.

An L-fuzzy subset of X is a mapping from X to L. The family of all L-subsets of X is denoted by $LF[X]$ (see [16]). When $L = [0, 1]$, the L-subsets of X are known as fuzzy subsets of X (see [17]). Let $\mu, \nu \in LF[X]$ be given, μ is said to be included in ν and written as $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in X$.

The following are the most popular operators on L-fuzzy sets: for all $\mu, \nu \in LF[X]$, $x \in X$, $(\mu \cup \nu)(x) = \mu(x) \vee \nu(x)$, $(\mu \cap \nu)(x) = \mu(x) \wedge \nu(x)$.

3 TL-Fuzzy Ideals

In this section, we shall introduce the notion of TL-fuzzy ideals in lattices and give some properties of them that will be used in the sequel.

Definition 3.1 Let $(X, \wedge, \vee, \leq)$ be a lattice and μ be an L-fuzzy subset of X. Then μ is called a TL-fuzzy ideal of X if it satisfies the following conditions: for all $x, y \in X$,

(i) $\mu(x \vee y) \geq \mu(x)T\mu(y)$,

(ii) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

We shall denote the set of all TL-fuzzy ideals of the lattice X as $TLFI[X]$.

In particular, a TL-fuzzy ideal is called an L-fuzzy ideal when $T = \wedge$. Moreover, when $L = [0, 1]$, a TL-fuzzy ideal and an L-fuzzy ideal of X are, respectively, referred to as a T-fuzzy ideal and fuzzy ideal of the lattice X.

Now, the following result gives an equivalent version of the concept of TL-fuzzy ideals in lattices.

Theorem 3.2 Let $(X, \wedge, \vee, \leq)$ be a lattice and μ be an L-fuzzy subset of X. Then μ is a TL-fuzzy ideal of X if and only if it satisfies the following conditions: for all $x, y \in X$,

(i) $\mu(x \vee y) \geq \mu(x)T\mu(y)$,

(ii) if $y \leq x$, then $\mu(y) \geq \mu(x)$.

Proof The proof is straightforward.

Example 3.3 (1) Let $X = \{0, a, b, c, 1\}$ be a lattice, the partial order on X is defined as shown in Fig. 1. Let $L = \{1, 2, 3, 4, 5, 6\}$, the partial order on L is defined as shown in Fig. 2, $T = \wedge$. Define two L-fuzzy subsets μ and ν of X as follows: $\mu(0) = 6$, $\mu(a) = 2$, $\mu(b) = 5$, $\mu(c) = 2$, $\mu(1) = 2$. By routine calculations, it is easy to check that μ is a TL-fuzzy ideal of X.

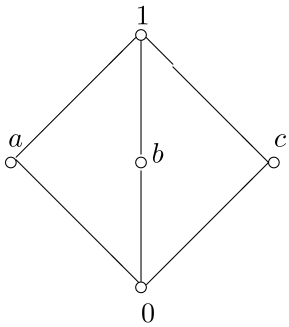

Fig. 1 The lattice X in Example 3.3(1)

Fig. 2 The lattice L in Example 3.3(1)

(2) Let $X = \{0, a, b, 1\}$ be a lattice, the partial order on X is defined as shown in Fig. 3. Let $L = [0, 1]$ and $T = T_L$, that is $xT_Ly = \max\{x+y-1, 0\}$, for any $x, y \in L = [0, 1]$. Define two L-fuzzy subsets μ and ν of X as follows: $\mu(0) = \frac{4}{5}$, $\mu(a) = \frac{3}{5}$, $\mu(b) = \frac{3}{10}$, $\mu(1) = \frac{3}{10}$. By routine calculations, it is easy to check that μ is a T_L-fuzzy ideal of X.

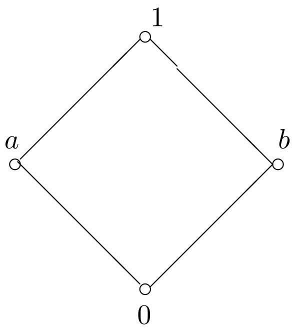

Fig. 3 The lattice X in Example 3.3(2)

Fig. 4 The lattice L in Example 4.3

In the following, we give some properties of TL-fuzzy ideals, which will be used in the sequel.

Proposition 3.4 Let $\mu_i (i \in I)$ be TL-fuzzy ideals of a lattice X. Then $\cap_{i\in I}\mu_i$ is a TL-fuzzy ideal of X.

Proof The proof is straightforward.

By the following example we show that the union of two TL-fuzzy ideals is not a TL-fuzzy ideal.

Example 3.5 Let $X = \{0, a, b, 1\}$ be a lattice, the partial order on X is defined as shown in Fig. 3 in Example 3.3 (2). Let $L = [0, 1]$ and $T = T_M$. Define two L-fuzzy subsets μ and ν of X as follows: $\mu(0) = \frac{3}{5}$, $\mu(a) = \frac{1}{2}$, $\mu(b) = \frac{1}{5}$, $\mu(1) = \frac{1}{5}$; $\nu(0) = \frac{7}{10}$, $\nu(a) = \frac{1}{10}$, $\nu(b) = \frac{3}{10}$, $\nu(1) = \frac{1}{10}$. Then we can check that both μ and ν are TL-fuzzy ideals of X. Now $(\mu\cup\nu)(0) = \frac{7}{10}$, $(\mu\cup\nu)(a) = \frac{1}{2}$, $(\mu\cup\nu)(b) = \frac{3}{10}$, $(\mu\cup\nu)(1) = \frac{1}{5}$. Since $(\mu\cup\nu)(a\vee b) = (\mu\cup\nu)(1) = \frac{1}{5} < (\mu\cup\nu)(a)\wedge(\mu\cup\nu)(b) = \frac{3}{10}$, $\mu\cup\nu$ is not a TL-fuzzy ideal of X.

4 The Lattice of TL-Fuzzy Ideals

Now, we give a procedure to construct the TL-fuzzy ideal generated by an L-fuzzy subset. And we shall discuss the algebraic structure of the set of all TL-fuzzy ideals in lattices.

Definition 4.1 Let μ be an L-fuzzy subset in a lattice X. A TL-fuzzy ideal ν of the lattice X is said to be generated by μ, if $\mu \subseteq \nu$ and for any TL-fuzzy ideal ω of X, $\mu \subseteq \omega$ implies $\nu \subseteq \omega$. The TL-fuzzy ideal generated by μ will be denoted by $(\mu]_{TL}$.

It follows from Definition 4.1 that $(\mu]_{TL}$ is the smallest TL-fuzzy ideal of the lattice X containing μ. And we can easily get that $(\mu]_{TL} = \cap_{i\in I}\{\mu_i \in TLFI[X]|\mu_i \supseteq \mu, i \in I\}$.

It is easy to verify that for μ and ν be L-fuzzy subsets of X. Then

(1) if μ is a TL-fuzzy ideal of the lattice X, then $(\mu]_{TL} = \mu$,

(2) $\mu \subseteq \nu$ implies $(\mu]_{TL} \subseteq (\nu]_{TL}$.

In what follows, we give the formula for calculating the TL-fuzzy ideals generated by L-fuzzy subsets.

Theorem 4.2 Let μ be an L-fuzzy subset in the lattice X. Then for any $x \in X$,

$$(\mu]_{TL}(x) = \vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)|x \leq a_1 \vee a_2 \vee \cdots \vee a_n, a_i \in X, i \in I\}.$$

Proof Let

$$\nu(x) = \vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)|x \leq a_1 \vee a_2 \vee \cdots \vee a_n, a_i \in X, i \in I\}.$$

First, we prove that ν is a TL-fuzzy ideal of X.

For any $x, y \in X$, we have that

$$\begin{aligned}
\nu(x)T\nu(y) &= (\vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)|x \leq a_1 \vee a_2 \vee \cdots \vee a_n, a_i \in X, i \in I\})T \\
&\quad (\vee\{\mu(b_1)T\mu(b_2)T\cdots T\mu(b_m)|y \leq b_1 \vee b_2 \vee \cdots \vee b_m, b_j \in X, j \in I\}) \\
&= \vee\{\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n)T\mu(b_1)T\mu(b_2)T\cdots T\mu(b_m)|x \\
&\leq a_1 \vee a_2 \vee \cdots \vee a_n, y \leq b_1 \vee b_2 \vee \cdots \vee b_m, a_i, b_j \in X, i, j \in I\} \\
&\leq \vee\{\mu(c_1)T\mu(c_2)T\cdots T\mu(c_l)|x \vee y \leq c_1 \vee c_2 \vee \cdots \vee c_l, c_i \in X, l \in I\} \\
&= \nu(x \vee y).
\end{aligned}$$

If $y \leq x$ and $x \leq a_1 \vee a_2 \vee \cdots \vee a_n$ for some $a_1, a_2, \cdots, a_n \in X$, then $y \leq a_1 \vee a_2 \vee \cdots \vee a_n$. It follows that $\nu(y) \geq \nu(x)$. By Theorem 3.2, we can get that ν is a TL-fuzzy ideal of X.

Next, since $x \leq x$, we have that $\nu(x) \geq \mu(x)$. So $\mu \subseteq \nu$.

Finally, suppose that ω is a TL-fuzzy ideal of X with $\mu \subseteq \omega$. Then for any $x \in X$, $\mu(x) \leq \omega(x)$. Moreover, for any $a_1, a_2, \cdots a_n \in X$ with $x \leq a_1 \vee a_2 \vee \cdots \vee a_n$, we have $\mu(a_1)T\mu(a_2)T\cdots T\mu(a_n) \leq \omega(a_1)T\omega(a_2)T\cdots T\omega(a_n) \leq \omega(a_1 \vee a_2 \vee \cdots \vee a_n) \leq \omega(x)$, since ω is a TL-fuzzy ideal of X. It follows that $\nu(x) \leq \omega(x)$. Thus, $\nu \subseteq \omega$.

Summarizing the above facts, we obtain that ν is the smallest TL-fuzzy ideal in the lattice X with $\mu \subseteq \nu$, that is, ν is the TL-fuzzy ideal generated by μ in X. Therefore, $\nu = (\mu]_{TL}$.

Example 4.3 Let $X = \{0, a, b, 1\}$ be a lattice, the partial order on X is defined as shown in Fig. 3 in Example 3.3 (2). Let $L = \{1, 2, 3, 4\}$, the partial order on L is defined as shown in Fig. 4 and $T = \wedge$. Define an L-fuzzy subset μ of X as follows: $\mu(0) = 3$, $\mu(a) = 2$, $\mu(b) = 4$, $\mu(1) = 1$. One can easily check that the TL-fuzzy ideal $(\mu]_{TL}$ generated by μ as follows: $(\mu]_{TL}(0) = 4$, $(\mu]_{TL}(a) = 2$, $(\mu]_{TL}(b) = 4$, $(\mu]_{TL}(1) = 2$.

Let X be a lattice. For any $\mu_1, \mu_2 \in TLFI[X]$, we define $\mu_1 \oplus \mu_2$ and $\mu_1 \otimes \mu_2$ as follows: $\mu_1 \oplus \mu_2 = \mu_1 \cap \mu_2$, $\mu_1 \otimes \mu_2 = \cap\{\mu \in TLFI[X] | \mu \supseteq \mu_1 \cup \mu_2\} = (\mu_1 \cup \mu_2]_{TL}$. In general, for any $\mu_i \in TLFI[X]$, where $i \in I$, we define $\oplus\{\mu_i | i \in I\} = \cap\{\mu_i | i \in I\}$, $\otimes\{\mu_i | i \in I\} = \cap\{\mu \in TLFI[X] | \mu \supseteq \cup_{i\in I}\mu_i\} = (\cup_{i\in I}\mu_i]_{TL}$. Therefore, we can get the following result.

Theorem 4.4 $(TLFI[X], \oplus, \otimes)$ is a complete lattice, which is called the lattice of TL-fuzzy ideals.

In particular, when $L = [0, 1]$, we can give the simple formulas for calculating the T-fuzzy ideals generated by the union of T-fuzzy ideals.

Theorem 4.5 Let μ_1, μ_2 be T-fuzzy ideals of a lattice X. Then for any $x \in X$,

$$(\mu_1 \cup \mu_2]_T(x) = \sup\{\{\mu_1(a) | x \le a\} \cup \{\mu_2(b) | x \le b\} \cup \{\mu_1(a)T\mu_2(b) | x \le a \vee b\}\}.$$

Proof By Theorem 4.2, we have

$$\begin{aligned}(\mu_1 \cup \mu_2]_T(x) &= \sup\{(\mu_1 \cup \mu_2)(a_1)T(\mu_1 \cup \mu_2)(a_2)T\cdots T(\mu_1 \cup \mu_2)(a_n)|x \\ &\le a_1 \vee \cdots \vee a_n, a_i \in X, i \in I\}.\end{aligned}$$

Given an arbitrary small $\epsilon > 0$, we have the following three cases:

Case 1 There exist $a_1, \cdots, a_n \in X$, satisfying

(1) $x \le a_1 \vee a_2 \vee \cdots \vee a_n$,

(2) $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \max\{\mu_1(a_1), \mu_2(a_1)\}T\cdots T\max\{\mu_1(a_n), \mu_2(a_n)\}$,

(3) $\mu_2(a_i) \le \mu_1(a_i), i = 1, \cdots, n$.

Thus $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_1(a_1)T\mu_1(a_2)\cdots T\mu_1(a_n)$.

Denote $a = a_1 \vee \cdots \vee a_n$. Since μ_1 is a T-fuzzy ideal of a lattice X, we have

$$\mu_1(a) \ge \mu_1(a_1)T\mu_1(a_2)\cdots T\mu_1(a_n),$$

it follows that $x \le a$ and $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_1(a)$.

Case 2 There exist $b_1, \cdots, b_m \in X$, satisfying

(1) $x \le b_1 \vee b_2 \vee \cdots \vee b_m$,

(2) $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \max\{\mu_1(b_1), \mu_2(b_1)\}T\cdots T\max\{\mu_1(b_m), \mu_2(b_m)\}$,

(3) $\mu_1(b_i) \le \mu_2(b_i), i = 1, \cdots, m$.

Thus $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_2(b_1)T\mu_2(b_2)T\cdots T\mu_2(b_m)$.

Denote $b = b_1 \vee b_2 \vee \cdots \vee b_m$. Since μ_2 is a T-fuzzy ideal of a lattice X, we have $\mu_2(b) \ge \mu_2(b_1)T\mu_2(b_2)\cdots T\mu_2(b_m)$, it follows that $x \le b$ and $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_2(b)$.

Cases 3 There exist $a_1, \cdots, a_s, b_1, \cdots, b_t \in X$, satisfying

(1) $x \le a_1 \vee \cdots \vee a_s \vee b_1 \vee \cdots \vee b_t$,
(2)

$$\begin{aligned}(\mu_1 \cup \mu_2]_T(x) \quad < \quad & \epsilon + \max\{\mu_1(a_1), \mu_2(a_1)\}T \cdots T \max\{\mu_1(a_s), \mu_2(a_s)\} \\ & T \max\{\mu_1(b_1), \mu_2(b_1)\}T \cdots \max\{\mu_1(b_t), \mu_2(b_t)\},\end{aligned}$$

(3) $\mu_2(a_i) \le \mu_1(a_i), \mu_1(b_j) \le \mu_2(b_j), i = 1, 2, \cdots, s, j = 1, 2, \cdots t.$
Thus

$$(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_1(a_1)T\mu_1(a_2)\cdots T\mu_1(a_s)T\mu_2(b_1)T\mu_2(b_2)T\cdots T\mu_2(b_t).$$

Denote $a = a_1 \vee \cdots \vee a_s$ and $b = b_1 \vee \cdots \vee b_t$, then $x \le a \vee b$. Since μ_1 and μ_2 are T-fuzzy ideals of a lattice X, we have

$$\mu_1(a) \ge \mu_1(a_1)T\mu_1(a_2)\cdots T\mu_1(a_s),\ \mu_2(b) \ge \mu_2(b_1)T\mu_2(b_2)\cdots T\mu_2(b_t).$$

It follows that $x \le a \vee b$ and $(\mu_1 \cup \mu_2]_T(x) < \epsilon + \mu_1(a)T\mu_2(b)$. Summarizing the above results we obtain

$$(\mu_1 \cup \mu_2]_T(x) \le \sup\{\{\mu_1(a)|x \le a\} \cup \{\mu_2(b)|x \le b\} \cup \{\mu_1(a)T\mu_2(b)|x \le a \vee b\}\}.$$

Conversely,

$$\begin{aligned} & \sup\{\mu_1(a)|x \le a\} \\ \le\ & \sup\{\max\{\mu_1(a), \mu_2(a)\}|x \le a\} \\ \le\ & \sup\{\max\{\mu_1(a_1), \mu_2(a_1)\}T \cdots T \max\{\mu_1(a_n), \mu_2(a_n)\}|x \le a_1 \vee \cdots \vee a_n\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned} & \sup\{\mu_2(b)|x \le b\} \\ \le\ & \sup\{\max\{\mu_1(a_1), \mu_2(a_1)\}T \cdots T \max\{\mu_1(a_n), \mu_2(a_n)\}|x \le a_1 \vee \cdots \vee a_n\}.\end{aligned}$$

Since $\mu_1(a)T\mu_2(b) \le \max\{\mu_1(a), \mu_2(a)\}T \max\{\mu_1(b), \mu_2(b)\}$, we have

$$\begin{aligned} & \sup\{\mu_1(a)T\mu_2(b)|x \le a \vee b\} \\ \le\ & \sup\{\max\{\mu_1(a), \mu_2(a)\}T \max\{\mu_1(b), \mu_2(b)\}|x \le a \vee b\} \\ \le\ & \sup\{\max\{\mu_1(a_1), \mu_2(a_1)\}T \cdots T \max\{\mu_1(a_n), \mu_2(a_n)\}|x \le a_1 \vee \cdots \vee a_n\}.\end{aligned}$$

Therefore, $\sup\{\{\mu_1(a)|x \le a\} \cup \{\mu_2(b)|x \le b\} \cup \{\mu_1(a)T\mu_2(b)|x \le a \vee b\}\} \le (\mu_1 \cup \mu_2]_T(x)$. This completes the proof.

When $L = [0, 1]$ and $T = T_M$, that is, $xT_My = \min\{x, y\}$ for all $x, y \in [0, 1]$, we can obtain the following main result.

Theorem 4.6 $(TLFI[X], \oplus, \otimes)$ is a complete modular lattice if X is a modular lattice.

Proof From Theorem 4.4, we have that $(TLFI[X], \oplus, \otimes)$ is a complete lattice. To verify that $(TLFI[X], \oplus, \otimes)$ is a modular lattice, we should show that it satisfies modular law. Now, assume $\mu_1, \mu_2, \mu_3 \in TLFI[X]$, where $\mu_1 \supseteq \mu_2$ and $x \in X$, the inequality $\mu_1 \oplus (\mu_2 \otimes \mu_3) \supseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$ is trivial. We only need to prove that $\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$.

Given an arbitrarily small $\epsilon > 0$, by Theorem 4.5, we have the following three cases:

Case 1 There exists $a \in X$ such that $x \leq a$ and $(\mu_2 \otimes \mu_3)(x) < \epsilon + \mu_2(a)$. And so

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_2(a)\}.$$

Since $\mu_2(a) \leq \mu_2(x) \leq \mu_1(x)$, it follows that $\min\{\mu_1(x), \mu_2(a)\} = \mu_2(a)$, hence

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \mu_2(a).$$

Combining $x \leq a$, we obtain

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \mu_2(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).$$

Case 2 There exists $b \in X$ such that $x \leq b$ and $(\mu_1 \otimes \mu_3)(x) < \epsilon + \mu_3(b)$. From $x \leq b$ it follows that $\mu_3(b) \leq \mu_3(x)$, hence

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_3(x)\}.$$

Combining $x \leq x$ and the definition of $\mu_2 \otimes (\mu_1 \oplus \mu_3)$, we have

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).$$

Case 3 There are $a, b \in X$ such that $x \leq a \vee b$ and $(\mu_2 \otimes \mu_3)(x) < \epsilon + \min\{\mu_2(a), \mu_3(b)\}$. Hence $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + \min\{\mu_1(x), \mu_2(a), \mu_3(b)\}$. Denote $b_1 = (x \vee a) \wedge b$, then $b_1 \leq x \vee a$ and $b_1 \leq b$. Notice that X is a modular lattice, $a \vee b_1 = a \vee ((x \vee a) \wedge b) = (x \vee a) \wedge (a \vee b) \geq x$. Since μ_1, μ_3 are fuzzy ideals, we have $\mu_3(b) \leq \mu_3(b_1)$ and $\mu_1(x \vee a) \leq \mu_1(b_1)$. It follows that

$$\begin{aligned}
& (\mu_1 \oplus_1 (\mu_2 \otimes \mu_3))(x) \\
\leq\ & \epsilon + \min\{\mu_2(a), \mu_1(a), \mu_1(x), \mu_3(b)\} \\
=\ & \epsilon + \min\{\mu_2(a), \min\{\mu_1(a), \mu_1(x)\}, \mu_3(b)\} \\
\leq\ & \epsilon + \min\{\mu_2(a), \mu_1(x \vee a), \mu_3(b)\} \\
\leq\ & \epsilon + \min\{\mu_2(a), \mu_1(b_1), \mu_3(b_1)\} \\
=\ & \epsilon + \min\{\mu_2(a), \min\{\mu_1(b_1), \mu_3(b_1)\}\} \\
=\ & \epsilon + \min\{\mu_2(a), (\mu_1 \oplus \mu_3)(b_1)\} \\
\leq\ & \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).
\end{aligned}$$

Since ϵ is arbitrary, we have $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x)$. Therefore,

$$\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3).$$

Summarizing the above facts, we get that for any $x \in X$ and given an arbitrarily small $\epsilon > 0$,

$$(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq \epsilon + (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x).$$

Therefore, $(\mu_1 \oplus (\mu_2 \otimes \mu_3))(x) \leq (\mu_2 \otimes (\mu_1 \oplus \mu_3))(x)$. So $\mu_1 \oplus (\mu_2 \otimes \mu_3) \subseteq \mu_2 \otimes (\mu_1 \oplus \mu_3)$, that is, $(TLFI[X], \oplus, \otimes)$ is a complete modular lattice.

5 T-Product of TL-Fuzzy Ideals

In this section, as a continuation of the work [11], we will explore necessary and sufficient conditions for a fuzzy ideal of a Cartesian product of lattices to be a T-product of fuzzy ideals of lattices under a left continuous t-norm T on a complete lattice L.

First, let us recall the Cartesian product of lattices for the sake of completeness.

Let $(X_1, \wedge_1, \vee_1, \leq_1)$ and $(X_2, \wedge_2, \vee_2, \leq_2)$ be two lattices. Define two binary operations $\wedge$ and $\vee$ on $X_1 \times X_2$ as follows: for any (x_1, x_2), $(y_1, y_2) \in X_1 \times X_2$,

$$(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge_1 y_1, x_2 \wedge_2 y_2), (x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_1 y_1, x_2 \vee_2 y_2).$$

Then $X_1 \times X_2$ is a lattice, which is called the Cartesian product lattice of X_1 and X_2. The corresponding partial order $\leq$ on $X_1 \times X_2$ as follows:

$$(x_1, x_2) \leq (y_1, y_2) \Longleftrightarrow x_1 \leq_1 y_1, x_2 \leq_2 y_2.$$

Definition 5.1 Let $\mu_i \in LF[X_i]$, $i = 1, 2$. Then the T-product of μ_i $(i = 1, 2)$ denoted by $\mu_1 \times_T \mu_2$ is defined as the L-fuzzy subset of $X_1 \times X_2$ that satisfies: for any $(x_1, x_2) \in X_1 \times X_2$, $\mu_1 \times_T \mu_2(x_1, x_2) = \mu_1(x_1)T\mu_2(x_2)$.

Theorem 5.2 Let μ_i be a TL-fuzzy ideal of a lattice X_i, $i = 1, 2$. Then $\mu_1 \times_T \mu_2$ is a TL-fuzzy ideal of $X_1 \times X_2$.

Proof Assume that μ_i be a TL-fuzzy ideal of a lattice X_i, $i = 1, 2$. For any (x_1, x_2), $(y_1, y_2) \in X_1 \times X_2$, then we have that

$$\begin{aligned}
& \mu_1 \times_T \mu_2((x_1, x_2) \vee (y_1, y_2)) = \mu_1 \times_T \mu_2((x_1 \vee_1 y_1, x_2 \vee_2 y_2)) \\
= \ & \mu_1(x_1 \vee_1 y_1)T\mu_2(x_2 \vee_2 y_2) \geq (\mu_1(x_1)T\mu_1(y_1))T(\mu_2(x_2)T\mu_2(y_2)) \\
= \ & (\mu_1(x_1)T\mu_2(x_2))T(\mu_1(y_1)T\mu_2(y_2)) \\
= \ & (\mu_1 \times_T \mu_2)(x_1, x_2)T(\mu_1 \times_T \mu_2)(y_1, y_2).
\end{aligned}$$

On the other hand, if $(x_1, x_2) \leq (y_1, y_2)$, that is $x_1 \leq_1 y_1, x_2 \leq_2 y_2$. Then we have

$$\mu_1 \times_T \mu_2(x_1, x_2) = \mu_1(x_1)T\mu_2(x_2) \geq \mu_1(y_1)T\mu_2(y_2) = \mu_1 \times_T \mu_2(y_1, y_2).$$

Therefore, $\mu_1 \times_T \mu_2$ is a TL-fuzzy ideal of $X_1 \times X_2$.

In what follows, we introduce the concepts of the projection and the cut shadow of an L-fuzzy set that are instrumental to determine necessary and sufficient conditions under t-norm operation.

Definition 5.3 Let $\mu \in LF[X_1 \times X_2]$. Then the projection of μ on X_i $(i = 1, 2)$ denoted by μ_{X_i} is defined as the L-fuzzy subset of X_i $(i = 1, 2)$ that satisfies, respectively, $\mu_{X_1}(x) = \bigvee_{b\in X_2} \mu(x, b)$ for any $x \in X_1$ and $\mu_{X_2}(y) = \bigvee_{a\in X_1} \mu(a, y)$ for any $y \in X_2$.

Theorem 5.4 Let X_1 and X_2 be two lattices and μ be a TL-fuzzy ideal of $X_1 \times X_2$. Then μ_{X_i} is a TL-fuzzy ideal of X_i, $i = 1, 2$.

Proof Assume that μ be a TL-fuzzy ideal of a lattice $X_1 \times X_2$. For any $x, y \in X_1$, then we have that

$$\begin{aligned}
&\mu_{X_1}(x \vee_1 y) = \bigvee_{b\in X_2} \mu(x \vee_1 y, b) \geq \bigvee_{b_1,b_2\in X_2} \mu(x \vee_1 y, b_1 \vee_2 b_2)\\
= &\bigvee_{b_1,b_2\in X_2} \mu((x, b_1) \vee (y, b_2)) \geq \bigvee_{b_1,b_2\in X_2} [\mu(x, b_1)T\mu(y, b_2)]\\
= &(\bigvee_{b_1\in X_2} \mu(x, b_1))T(\bigvee_{b_2\in X_2} \mu(y, b_2)) = \mu_{X_1}(x)T\mu_{X_1}(y).
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
&\mu_{X_1}(x \wedge_1 y) = \bigvee_{b\in X_2} \mu(x \wedge_1 y, b) \geq \bigvee_{b_1,b_2\in X_2} \mu(x \wedge_1 y, b_1 \wedge_2 b_2)\\
= &\bigvee_{b_1,b_2\in X_2} \mu((x, b_1) \wedge (y, b_2)) \geq \bigvee_{b_1,b_2\in X_2} [\mu(x, b_1) \vee \mu(y, b_2)]\\
= &(\bigvee_{b_1\in X_2} \mu(x, b_1)) \vee (\bigvee_{b_2\in X_2} \mu(y, b_2)) = \mu_{X_1}(x) \vee \mu_{X_1}(y).
\end{aligned}$$

Therefore, μ_{X_1} is a TL-fuzzy ideal of X_1. Dually, we have that μ_{X_2} is a TL-fuzzy ideal of X_2.

Definition 5.5 Let $\mu \in LF[X_1 \times X_2]$ and $a \in X_1$, $b \in X_2$. Then the cut shadow of μ with respect to b denoted by $\mu_1|_b$ is defined as the L-fuzzy subset of X_1 that satisfies: for any $x \in X_1$, $\mu_1|_b(x) = \mu(x, b)$. Similarly, the cut shadow of μ with respect to a denoted by $\mu_2|_a$ is defined as the L-fuzzy subset of X_2 that satisfies: for any $y \in X_2$, $\mu_2|_a(y) = \mu(a, y)$.

Theorem 5.6 Let X_1 and X_2 be two lattices and μ be a TL-fuzzy ideal of $X_1 \times X_2$, let $a \in X_1$, $b \in X_2$. Then $\mu_1|_b$ is a TL-fuzzy ideal of X_1 and $\mu_2|_a$ is a TL-fuzzy ideal of X_2.

Proof The proof is straightforward.

In order to obtain necessary and sufficient conditions for a TL-fuzzy ideal of a Cartesian product lattice to be a T-product of fuzzy ideals of lattices, we give the following lemma.

Lemma 5.7 Let X_1 and X_2 be two lattices and μ be a TL-fuzzy ideal of $X_1 \times X_2$ such that $Im\mu \subseteq D_T$, let $a \in X_1$, $b \in X_2$. Then $\mu_1|_b \times_T \mu_2|_a \subseteq \mu \subseteq \mu_{X_1} \times_T \mu_{X_2}$.

Proof Assume that μ is a TL-fuzzy ideal of a lattice $X_1 \times X_2$.

First, we prove that $\mu \subseteq \mu_{X_1} \times_T \mu_{X_2}$. For any $(x, y) \in X_1 \times X_2$, we have that

$$\mu(x, y) \leq \bigvee_{b\in X_2} \mu(x, b) = \mu_{X_1}(x)$$

and $\mu(x, y) \leq \bigvee_{a\in X_1} \mu(a, y) = \mu_{X_2}(y)$.

Thus, we obtain $\mu(x,y)T\mu(x,y) \le \mu_{X_1}(x)T\mu_{X_2}(y)$, then $\mu(x,y) \le \mu_{X_1} \times_T \mu_{X_2}(x,y)$. Hence, $\mu \subseteq \mu_{X_1} \times_T \mu_{X_2}$.

Next, let us check $\mu_1|_b \times_T \mu_2|_a \subseteq \mu$. For any $(x,y) \in X_1 \times X_2$, we can have

$$\begin{aligned} & \mu_1|_b \times_T \mu_2|_a(x,y) = \mu_1|_b(x)T\mu_2|_a(y) = \mu(x,b)T\mu(a,y) \le \mu((x,b) \vee (a,y)) \\ = \; & \mu((x \vee_1 a, b \vee_2 y)) = \mu((x,y) \vee (a,b)) \le \mu(x,y). \end{aligned}$$

Thus $\mu_1|_b \times_T \mu_2|_a \subseteq \mu$. Combining the above arguments, we can obtain

$$\mu_1|_b \times_T \mu_2|_a \subseteq \mu \subseteq \mu_{X_1} \times_T \mu_{X_2}$$

for all $a \in X_1$, $b \in X_2$.

The following theorem gives one of the main results in this paper.

Theorem 5.8 Let X_1 and X_2 be two lattices with the bottom element 0 and μ be a TL-fuzzy ideal of $X_1 \times X_2$ such that $Im\mu \subseteq D_T$. Then μ is the T-product of a TL-fuzzy ideal of X_1 and a TL-fuzzy ideal of X_2 if and only if $\mu_1|_0 \times_T \mu_2|_0 = \mu_{X_1} \times_T \mu_{X_2}$.

Proof Assume that $\mu = \mu_1 \times_T \mu_2$, where μ_1 and μ_2 are TL-fuzzy ideals of X_1 and X_2, respectively. Then $\mu_1(x) \le \mu_1(0)$ for any $x \in X_1$ and $\mu_2(y) \le \mu_2(0)$ for any $y \in X_2$. Thus we have $\bigvee_{x\in X_1} \mu_1(x) = \mu_1(0)$ and $\bigvee_{y\in X_2} \mu_2(y) = \mu_2(0)$. Notice this, we can obtain that

$$\begin{aligned} & \mu_1|_0(x) = \mu(x,0) = \mu_1 \times_T \mu_2(x,0) = \mu_1(x)T\mu_2(0) = \mu_1(x)T(\bigvee_{y\in X_2} \mu_2(y)) \\ = \; & \bigvee_{y\in X_2} [\mu_1(x)T\mu_2(y)] = \bigvee_{y\in X_2} \mu(x,y) = \mu_{X_1}(x). \end{aligned}$$

Hence, $\mu_1|_0 = \mu_{X_1}$. Similarly, we can get that $\mu_2|_0 = \mu_{X_2}$. Therefore,

$$\mu_1|_0 \times_T \mu_2|_0 = \mu_{X_1} \times_T \mu_{X_2}.$$

Conversely, assume that $\mu_1|_0 \times_T \mu_2|_0 = \mu_{X_1} \times_T \mu_{X_2}$. By Lemma 5.7, we can get $\mu = \mu_{X_1} \times_T \mu_{X_2}$. Since μ is a TL-fuzzy ideal of $X_1 \times X_2$, it follows from Theorem 5.4, we have μ_{X_i} is a TL-fuzzy ideal of X_i, $i = 1, 2$. That is, μ is the T-product of a TL-fuzzy ideal of X_1 and a TL-fuzzy ideal of X_2.

Open Problem Whether the lattice of all TL-fuzzy ideals of a lattice X forms a distributive lattice or even a modular lattice.

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格的TL - 模糊理想

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摘 要: 本文研究了格的TL - 模糊理想. 利用生成TL - 模糊理想, 证明了一个模格的全体T_M - 模糊理想形成一个完备的模格. 此外, 利用L - 模糊集的投影和截影, 获得了将直积格的TL - 模糊理想表示成分量格的TL - 模糊理想的T - 直积的一个充分必要条件. 所得结果进一步推广和发展了格的模糊理想的理论.

关键词: 左连续t - 模; 模糊理想; 模格; T - 直积

MR(2010)**主题分类号**: 06D72; 08A72 **中图分类号**: O159