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SOME COMMON FIXED POINT THEOREMS FOR MAPPINGS ON CONE b-METRIC SPACES

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Abstract: The common fixed point problem of mappings is studied in this article. Some existence and uniqueness of points of coincidence and common fixed points for four mappings are obtained in cone-b metric spaces by the method of successive approximation. Some related results in cone metric spaces are generalized to cone b-metric spaces. An example is given to support our results.

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1 Introduction

As a important theory in mathematics, the fixed point theory was studied extensively since 1922 with the well-known Banach contractive mapping principle. This principle is a forceful tool in solving many existence problems in mathematical sciences and engineering. As a fundamental result in fixed theory, it is extended by several authors on different metric spaces. In 2007, by replacing the set of real numbers in an ordered Banach space, Huang and Zhang [1] defined the concept of cone metric space, the class of which is effectively larger than that of the metric spaces, and they proved some fixed point theorems for mappings satisfying certain contractive conditions on cone metric spaces. Rezapour and Hamlbarani [2] obtained some generalizations of the results in [1] by omitting the assumption of normal cone. After then, the fixed point theory has evolved speedily in cone metric spaces, many researchers were motivated to study fixed theorems as well as common fixed point theorems for two or more mappings on cone metric spaces, some literatures on this subject exist. For details to see [3–15]. In 2011, Hussain and Shah [16] introduced the concept of cone b-metric spaces as a generalization of b-metric spaces and cone metric spaces. Several authors studied fixed point and common fixed point problems on cone b-metric spaces. For details to see [17–21]. In this paper, we shall show that some existence and uniqueness of points of

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coincidence and common fixed points for four mappings satisfying a Lipschitz type condition in non-normal cone b-metric spaces.

2 Preliminaries

Let E be a real Banach space, P a subset of E, and θ is the zero element of E, P is called to be a cone if

(i) P is closed, nonempty and $P \neq \{\theta\}$;

(ii) $a, b \in R$, $a, b \geq 0, x, y \in P$ imply that $ax + by \in P$;

(iii) $x \in P$ and $-x \in P$ imply that $x = \theta$.

We denote the the interior of P by int$P$, if int$P \neq \phi$, the cone P is called a solid.

Let P be a cone, a partial ordering "$\leq$" on E with respect to P can be defined as follows: for all $x, y \in E$, $x \leq y$ if and only if $y - x \in P$. While $x \ll y$ stands for $y - x \in \text{int}P$, we shall write $x < y$ to indicate that $x \leq y$ and $x \neq y$.

The cone P is called normal if there is a positive constant number N such that $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$ for all $x, y \in P$. The least positive number satisfying the above inequality is called the normal constant of P.

Definition 2.1 [8] Let X be a nonempty set and E a real Banach space equipped with the partial ordering "$\leq$" respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies following condition:

(i) $\theta \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, y) \leq s(d(x, z) + d(z, y))$ for all $x, y, z \in X$,

where s is a constant number and $s \geq 1$. Then d is called a cone b-metric on X with constant s and (X, d) is called a cone b-metric space.

Clearly, a cone metric space is a cone b-metric space with constant number $s = 1$, but a cone b-metric space with constant $s > 1$ may be not necessarily a cone metric space (see [16,19]). So the concept of the cone b-metric space is more general than that of the cone metric space.

Definition 2.2 [16] Let (X, d) be a cone b-metric space, $\{x_n\} \subset X$. We say $\{x_n\}$ is

(i) a Cauchy sequence if for every c in E with $\theta \ll c$, there is a positive integer number N such that $d(x_n, x_m) \ll c$ for all $n, m > N$;

(ii) a Convergent sequence if for every c in E with $\theta \ll c$, there is a positive integer number N such that $d(x_n, x) \ll c$ for all $n > N$ and some fixed x in X, we denote this by $x_n \to x$ $(n \to \infty)$.

A cone b-metric space$X$ is said to be complete if every Cauchy sequence in X is convergent in X .

The following properties are often used (note that the cone need not be normal).

Proposition 2.3 [16] Let P be a cone in the real Banach space E.

(i) Let $\{a_n\}$ be a sequence in E, and $\theta \leq a_n \to \theta$, then for every $c \in \text{int}P$, there exists positive integer number N such that $a_n \ll c$ for all $n > N$.

(ii) Let $a, b, c \in E$, $a \leq b$ and $b \ll c$, then $a \ll c$.

(iii) Let $u \in E$, and $\theta \leq u \ll c$ for each $c \in \text{int}P$, then $u = \theta$.

(iv) Let $a \in P$, $a \leq \lambda a$ and $0 < \lambda < 1$, then $a = \theta$.

Since the topology on a cone b-metric space is a Hausdorff topology (see [8]), the following proposition is clearly.

Proposition 2.4 The limit of a convergent sequence in a cone b-metric space is unique.

Definition 2.5 [3] Let f and g be self maps of a set X . If $w = fx = gx$ for some x in X , then x is called a coincidence point of f and g , and w is called a point of coincidence of f and g .

Definition 2.6 [5] The mappings $f, g : X \to X$ are weakly compatible if, for every $x \in X$, $fgx = ggx$ holds whenever $fx = gx$.

We say that $\{f, g\}$ is a weakly compatible pair.

Definition 2.7 [3] Let f and g be weakly compatible self maps of a set X . If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g .

3 Main Results

In this section, we shall show that some existence and uniqueness of points of coincidence and common fixed points for four mappings satisfying a Lipschitz type conditions in a cone b-metric space without the assumption of normality. We always suppose that P is a solid cone in E.

Theorem 3.1 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$\begin{aligned} d(Fx, Gy) \leq\ & a_1(x, y)d(Hx, Ty) + a_2(x, y)d(Hx, Fx) + a_3(x, y)d(Ty, Gy) \\ & + a_4(x, y)d(Hx, Gy) + a_5(x, y)d(Ty, Fx), \end{aligned} \tag{3.1}$$

where $a_i(x, y) : X \times X \to [0, +\infty)(i = 1, 2, 3, 4, 5)$ are nonnegative real functions which satisfy that

$$a_1(x, y) + a_4(x, y) + a_5(x, y) < 1, a_2(x, y) + a_5(x, y) < \frac{1}{s}, a_3(x, y) + a_4(x, y) < \frac{1}{s},$$

and

$$L_1 = \sup_{x,y \in X} \frac{a_1(x, y) + a_3(x, y) + sa_5(x, y)}{1 - a_2(x, y) - sa_5(x, y)} < +\infty;$$

$$L_2 = \sup_{x,y \in X} \frac{a_1(x, y) + a_2(x, y) + sa_4(x, y)}{1 - a_3(x, y) - sa_4(x, y)} < +\infty;$$

$$L_3 = \sup_{x,y \in X} \frac{sa_1(x, y) + a_2(x, y) + sa_4(x, y)}{1 - sa_3(x, y) - sa_4(x, y)} < +\infty;$$

$$L_4 = \sup_{x,y \in X} \frac{sa_1(x, y) + sa_3(x, y) + a_5(x, y)}{1 - sa_3(x, y) - sa_4(x, y)} < +\infty;$$

$$L_5 = \sup_{x,y\in X} \frac{sa_1(x,y)+a_3(x,y)+sa_5(x,y)}{1-sa_2(x,y)-sa_5(x,y)} < +\infty;$$
$$L_6 = \sup_{x,y\in X} \frac{sa_1(x,y)+sa_2(x,y)+a_4(x,y)}{1-sa_2(x,y)-sa_5(x,y)} < +\infty;$$
$$L_1L_2 < \frac{1}{s^2}.$$

If $F(X)\subseteq T(X), G(X)\subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F,H\}$ and $\{G,T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Let x_0 be an arbitrary point in X. Since $F(X)\subseteq T(X), G(X)\subseteq H(X)$, there exist $x_1, x_2\in X$ such that $Fx_0 = Tx_1, Gx_1 = Hx_2$. Continuing this process, we can obtain the two sequences $\{x_n\}, \{y_n\}\subset X$ such that

$$y_{2n} = Fx_{2n} = Tx_{2n+1}, y_{2n+1} = Gx_{2n+1} = Hx_{2n+2}, n = 0,1,2,\cdots. \tag{3.2}$$

From(3.1) and (3.2), we have

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) =& d(Fx_{2n+2},Gx_{2n+1})\\
\le& a_1(x_{2n+2},x_{2n+1})d(Hx_{2n+2},Tx_{2n+1}) + a_2(x_{2n+2},x_{2n+1})d(Hx_{2n+2},Fx_{2n+2})\\
&+ a_3(x_{2n+2},x_{2n+1})d(Tx_{2n+1},Gx_{2n+1}) + a_4(x_{2n+2},x_{2n+1})d(Hx_{2n+2},Gx_{2n+1})\\
&+ a_5(x_{2n+2},x_{2n+1})d(Tx_{2n+1},Fx_{2n+2})\\
=& a_1(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+1}) + a_2(x_{2n+2},x_{2n+1})d(y_{2n+1},y_{2n+2})\\
&+ a_3(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+1}) + a_4(x_{2n+2},x_{2n+1})d(y_{2n+1},y_{2n+1})\\
&+ a_5(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+2})\\
\le& a_1(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+1}) + a_2(x_{2n+2},x_{2n+1})d(y_{2n+1},y_{2n+2})\\
&+ a_3(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+1}) + sa_5(x_{2n+2},x_{2n+1})d(y_{2n},y_{2n+1})\\
&+ sa_5(x_{2n+2},x_{2n+1})d(y_{2n+1},y_{2n+2}),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &\le \frac{a_1(x_{2n+2},x_{2n+1})+a_3(x_{2n+2},x_{2n+1})+sa_5(x_{2n+2},x_{2n+1})}{1-a_2(x_{2n+2},x_{2n+1})-sa_5(x_{2n+2},x_{2n+1})}d(y_{2n},y_{2n+1})\\
&\le L_1d(y_{2n},y_{2n+1}).
\end{aligned} \tag{3.3}$$

Similarly, also we have

$$\begin{aligned}
d(y_{2n},y_{2n+1}) &\le \frac{a_1(x_{2n},x_{2n+1})+a_2(x_{2n},x_{2n+1})+sa_4(x_{2n},x_{2n+1})}{1-a_3(x_{2n},x_{2n+1})-sa_4(x_{2n},x_{2n+1})}d(y_{2n-1},y_{2n})\\
&\le L_2d(y_{2n-1},y_{2n}).
\end{aligned} \tag{3.4}$$

From(3.3), (3.4), we can obtain that

$$\begin{aligned}
d(y_{2n+1},y_{2n+2}) &\le L_1d(y_{2n},y_{2n+1}) \le L_1L_2d(y_{2n-1},y_{2n})\\
&\le \ldots \le L_1(L_1L_2)^nd(y_0,y_1),
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &\le L_2 d(y_{2n-1}, y_{2n}) \le L_1 L_2 d(y_{2n-2}, y_{2n-1}) \\ &\le \dots \le (L_1L_2)^n d(y_0, y_1) \end{aligned} \tag{3.6}$$

for all $n = 0, 1, 2, \cdots$. Since $L_1L_2 < \frac{1}{s^2} \le 1$, by (3.5), (3.6), we have

$$\begin{aligned} d(y_{2m}, y_{2n}) &\le sd(y_{2m}, y_{2m+1}) + sd(y_{2m+1}, y_{2n}) \\ &\le sd(y_{2m}, y_{2m+1}) + s^2 d(y_{2m+1}, y_{2m+2}) + s^3 d(y_{2m+2}, y_{2m+3}) + \dots \\ &\quad + s^{2n-2m-2} d(y_{2n-3}, y_{2n-2}) + s^{2n-2m-1} d(y_{2n-2}, y_{2n-1}) + s^{2n-2m-1} d(y_{2n-1}, y_{2n}) \\ &\le (s(L_1L_2)^m + s^2 L_1 (L_1L_2)^m + s^3 (L_1L_2)^{m+1} + \dots \\ &\quad + s^{2n-2m-2} L_1 (L_1L_2)^{n-2} + s^{2n-2m-1} (L_1L_2)^{n-1} + s^{2n-2m-1} L_1 (L_1L_2)^{n-1}) d(y_0, y_1) \\ &= (s(L_1L_2)^m \sum_{i=0}^{n-m-1} (s^2 L_1L_2)^i + s^2 L_1 (L_1L_2)^m \sum_{i=0}^{n-m-2} (s^2 L_1L_2)^i \\ &\quad + s^{-2m+1} L_1 (s^2 L_1L_2)^{n-1}) d(y_0, y_1) \\ &\le (\frac{s(L_1L_2)^m}{1 - s^2 L_1L_2} + \frac{s^2 L_1 (L_1L_2)^m}{1 - s^2 L_1L_2} + s^{-2m+1} L_1) d(y_0, y_1) \\ &= (\frac{s(L_1L_2)^m (1 + sL_1)}{1 - s^2 L_1L_2} + s^{-2m+1} L_1) d(y_0, y_1), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} d(y_{2m+1}, y_{2n+1}) &\le sd(y_{2m+1}, y_{2m+2}) + sd(y_{2m+2}, y_{2n+1}) \\ &\le sd(y_{2m+1}, y_{2m+2}) + s^2 d(y_{2m+2}, y_{2m+3}) + s^3 d(y_{2m+3}, y_{2m+4}) + \dots \\ &\quad + s^{2n-2m-2} d(y_{2n-2}, y_{2n-1}) + s^{2n-2m-1} d(y_{2n-1}, y_{2n}) + s^{2n-2m-1} d(y_{2n}, y_{2n+1}) \\ &\le (sL_1 (L_1L_2)^m + s^2 (L_1L_2)^{m+1} + s^3 L_1 (L_1L_2)^{m+1} + \dots \\ &\quad + s^{2n-2m-2} (L_1L_2)^{n-1} + s^{2n-2m-1} L_1 (L_1L_2)^{n-1} + s^{2n-2m-1} (L_1L_2)^n) d(y_0, y_1) \\ &= (sL_1 (L_1L_2)^m \sum_{i=0}^{n-m-1} (s^2 L_1L_2)^i + s^2 (L_1L_2)^{m+1} \sum_{i=0}^{n-m-2} (s^2 L_1L_2)^i \\ &\quad + s^{-2m-1} (s^2 L_1L_2)^n) d(y_0, y_1) \\ &\le (\frac{sL_1 (L_1L_2)^m}{1 - s^2 L_1L_2} + \frac{s^2 (L_1L_2)^{m+1}}{1 - s^2 L_1L_2} + s^{-2m-1}) d(y_0, y_1) \\ &= (\frac{sL_1 (L_1L_2)^m (1 + sL_2)}{1 - s^2 L_1L_2} + s^{-2m-1}) d(y_0, y_1) \end{aligned} \tag{3.8}$$

for $n > m$. It is clearly that

$$\begin{aligned} &(\frac{s(L_1L_2)^m (1 + sL_1)}{1 - s^2 L_1L_2} + s^{-2m+1} L_1) d(y_0, y_1) \to \theta \ (n \to \infty), \\ &(\frac{sL_1 (L_1L_2)^m (1 + sL_2)}{1 - s^2 L_1L_2} + s^{-2m-1}) d(y_0, y_1) \to \theta \ (n \to \infty). \end{aligned}$$

By Proposition 2.1 (i), for each $\theta \ll c$, there exists a natural number N_1 such that

$$d(y_{2m}, y_{2n}) \leq (\frac{s(L_1L_2)^m(1+sL_1)}{1-s^2L_1L_2} + s^{-2m+1}L_1)d(y_0, y_1) \ll c,$$
$$d(y_{2m+1}, y_{2n+1}) \leq (\frac{sL_1(L_1L_2)^m(1+sL_2)}{1-s^2L_1L_2} + s^{-2m-1})d(y_0, y_1) \ll c$$

for all $n > N_1$. It implies that $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are a cauchy sequence.

If $T(X)$ is a complete subspace of X, since $\{y_{2n}\} \subset T(X)$ and $\{y_{2n}\}$ is a cauchy sequence, there exist $q \in T(X)$ and $p \in X$ such that $y_{2n} \to q(n \to \infty)$ and $q = Tp$. Then, from (3.1), we have

$$\begin{aligned} d(y_{2n}, Gp) =& d(Fx_{2n}, Gp) \\ \leq& a_1(x_{2n}, p)d(Hx_{2n}, Tp) + a_2(x_{2n}, p)d(Hx_{2n}, Fx_{2n}) + a_3(x_{2n}, p)d(Tp, Gp) \\ &+ a_4(x_{2n}, p)d(Hx_{2n}, Gp) + a_5(x_{2n}, p)d(Tp, Fx_{2n}) \\ =& a_1(x_{2n}, p)d(y_{2n-1}, q) + a_2(x_{2n}, p)d(y_{2n-1}, y_{2n}) + a_3(x_{2n}, p)d(q, Gp) \\ &+ a_4(x_{2n}, p)d(y_{2n-1}, Gp) + a_5(x_{2n}, p)d(y_{2n}, q) \\ \leq& sa_1(x_{2n}, p)d(y_{2n-1}, y_{2n}) + sa_1(x_{2n}, p)d(y_{2n}, q) + a_2(x_{2n}, p)d(y_{2n-1}, y_{2n}) \\ &+ sa_3(x_{2n}, p)d(y_{2n}, q) + sa_3(x_{2n}, p)d(y_{2n}, Gp) + sa_4(x_{2n}, p)d(y_{2n-1}, y_{2n}) \\ &+ sa_4(x_{2n}, p)d(y_{2n}, Gp) + a_5(x_{2n}, p)d(y_{2n}, q) \end{aligned}$$

for $n = 0, 1, 2, \cdots$, which implies that

$$\begin{aligned} d(y_{2n}, Gp) \leq& \frac{sa_1(x_{2n}, p) + a_2(x_{2n}, p) + sa_4(x_{2n}, p)}{1 - sa_3(x_{2n}, p) - sa_4(x_{2n}, p)} d(y_{2n-1}, y_{2n}) \\ &+ \frac{sa_1(x_{2n}, p) + sa_3(x_{2n}, p) + a_5(x_{2n}, p)}{1 - sa_3(x_{2n}, p) - sa_4(x_{2n}, p)} d(y_{2n}, q) \\ \leq& L_3 d(y_{2n-1}, y_{2n}) + L_4 d(y_{2n}, q) \end{aligned}$$

for $n = 0, 1, 2, \cdots$. Since $y_{2n} \to q(n \to \infty)$ and (3.5), for each $\theta \ll c$, there exists a positive integer number N_2 such that

$$L_3 d(y_{2n-1}, y_{2n}) \ll \frac{c}{2}, L_4 d(y_{2n}, q) \ll \frac{c}{2}$$

for all $n > N_2$. It implies that $d(y_{2n}, Gp) \ll c$ for all $n > N_2$, so $y_{2n} \to Gp$ $(n \to \infty)$. By Proposition 2.2, we have $Gp = q$. So, q is a point of coincidence of G and T.

Since $G(X) \subseteq H(X)$, there exists $u \in X$ such that $q = Gp = Hu$. By using (3.1) , we have

$$\begin{aligned} d(Fu, q) =& d(Fu, Gp) \\ \leq& a_1(u, p)d(Hu, Tp) + a_2(u, p)d(Hu, Fu) + a_3(u, p)d(Tp, Gp) \\ &+ a_4(u, p)d(Hu, Gp) + a_5(u, p)d(Tp, Fu) \\ =& (a_2(u, p) + a_5(u, p))d(Fu, q) = K_1 d(Fu, q), \end{aligned}$$

where $K_1 = a_2(u,p) + a_5(u,p) < \frac{1}{s} < 1$. So, $d(Fu,q) = \theta$ by Proposition 2.1 (iii), that is $Fu = q$. It implies that q is a point of coincidence of pairs of F and H. If q_1 is another point such that $Fu_1 = Hu_1 = q_1 = Gp_1 = Tp_1$ for some $u_1 \in X$ and $p_1 \in X$. By using (3.1) again

$$\begin{aligned} d(q_1,q) &= d(Fu_1,Gp) \\ &\leq a_1(u_1,p)d(Hu_1,Tp) + a_2(u_1,p)d(Hu_1,Fu_1) + a_3(u_1,p)d(Tp,Gp) \\ &\quad + a_4(u_1,p)d(Hu_1,Gp) + a_5(u_1,p)d(Tp,Fu_1) \\ &= (a_1(u_1,p) + a_4(u_1,p) + a_5(u_1,p))d(q_1,q) = K_2 d(q_1,q), \end{aligned} \tag{3.9}$$

where $K_2 = a_1(u_1,p) + a_4(u_1,p) + a_5(u_1,p) < 1$. It follows by Proposition 2.1 (iii) that $d(q_1,q) = \theta$, $q_1 = q$.

If $H(X)$ is a complete subspace of X, since $\{y_{2n+1}\} \subset H(X)$ and $\{y_{2n+1}\}$ is a cauchy sequence, there exist $w \in H(X)$ and $z \in X$ such that $y_{2n+1} \to w(n \to \infty)$ and $w = Hz$. By (3.1), we have

$$\begin{aligned} &d(y_{2n+1},Fz) = d(Fz,Gx_{2n+1}) \\ &\leq a_1(z,x_{2n+1})d(Hz,Tx_{2n+1}) + a_2(z,x_{2n+1})d(Hz,Fz) + a_3(z,x_{2n+1})d(Tx_{2n+1},Gx_{2n+1}) \\ &\quad + a_4(z,x_{2n+1})d(Hz,Gx_{2n+1}) + a_5(z,x_{2n+1})d(Tx_{2n+1},Fz) \\ &= a_1(z,x_{2n+1})d(w,y_{2n}) + a_2(z,x_{2n+1})d(w,Fz) + a_3(z,x_{2n+1})d(y_{2n},y_{2n+1}) \\ &\quad + a_4(z,x_{2n+1})d(w,y_{2n+1}) + a_5(z,x_{2n+1})d(y_{2n},Fz) \\ &\leq sa_1(z,x_{2n+1})d(w,y_{2n+1}) + sa_1(z,x_{2n+1})d(y_{2n},y_{2n+1}) + sa_2(z,x_{2n+1})d(w,y_{2n+1}) \\ &\quad + sa_2(z,x_{2n+1})d(y_{2n+1},Fz) + a_3(z,x_{2n+1})d(y_{2n},y_{2n+1}) + a_4(z,x_{2n+1})d(w,y_{2n+1}) \\ &\quad + sa_5(z,x_{2n+1})d(y_{2n},y_{2n+1}) + sa_5(z,x_{2n+1})d(y_{2n+1},Fz) \end{aligned}$$

for all $n = 0,1,2,\cdots$, which is equivalent to

$$\begin{aligned} d(y_{2n+1},Fz) \leq& \frac{sa_1(z,x_{2n+1}) + a_3(z,x_{2n+1}) + sa_5(z,x_{2n+1})}{1 - sa_2(z,x_{2n+1}) - sa_5(z,x_{2n+1})} d(y_{2n},y_{2n+1}) \\ &+ \frac{sa_1(z,x_{2n+1}) + sa_2(z,x_{2n+1}) + a_4(z,x_{2n+1})}{1 - sa_2(z,x_{2n+1}) - sa_5(z,x_{2n+1})} d(y_{2n+1},w) \\ \leq& L_5 d(y_{2n},y_{2n+1}) + L_6 d(y_{2n+1},q) \end{aligned}$$

for $n = 0,1,2,\cdots$. Since $y_{2n+1} \to w$ $(n \to \infty)$ and (3.6), for each $\theta \ll c$, there exists a natural number N_3 such that

$$L_5 d(y_{2n},y_{2n+1}) \ll \frac{c}{2}, L_6 d(y_{2n+1},q) \ll \frac{c}{2}$$

for all $n > N_3$. It implies that $d(y_{2n+1},Fz) \ll c$ for all $n > N_3$. So $y_{2n+1} \to Fz$ $(n \to \infty)$. By Proposition 2.2, we have $Fz = w$. Since $F(x) \subseteq T(X)$, there exists a point $v \in X$, such that $w = Fz = Tv$. Then

$$\begin{aligned} d(w, Gv) =& d(Fz, Gv) \\ \leq& a_1(z,v)d(Hz,Tv) + a_2(z,v)d(Hz,Fz) + a_3(z,v)d(Tv,Gv) \\ &+ a_4(z,v)d(Hz,Gv) + a_5(z,v)d(Tv,Fz) \\ =&(a_3(z,v) + a_4(z,v))d(w,Gv) = K_3 d(w,Gv), \end{aligned}$$

where $K_3 = a_3(z,v) + a_4(z,v) < \frac{1}{s} < 1$. By Proposition 2.1 (iii), we have $d(w, Gv) = \theta$, that is $Gv = w$. So w is a point of coincidence of F and H. As we do it in (3.9), one can prove that it is unique.

If $F(X)$ or $G(X)$ is a complete subspace of X, by the same arguments as above, also we can obtain the same result as the above. If $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, respectively, then F, G, H, T have an unique common fixed point by proposition 2.3. This complete the proof of theorem.

Corollary 3.2 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$d(Fx, Gy) \leq a_1 d(Hx, Ty) + a_2 d(Hx, Fx) + a_3 d(Ty, Gy) + a_4 d(Hx, Gy) + a_5 d(Ty, Fx).$$

where $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ are nonnegative real number which satisfy that

$$a_1 + a_4 + a_5 < \frac{1}{s}, a_2 + a_5 < \frac{1}{s}, a_3 + a_4 < \frac{1}{s},$$

and there exists $\delta > 0$ such that $a_1 + a_2 + a_3 + sa_4 + sa_5 = \frac{1+\delta}{s}$, and $(a_3 - a_2)(a_5 - a_4) > \frac{2\delta}{s^3}$. If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Take $a_i(x, y) = a_i$ $(i = 1, 2, 3, 4, 5)$. Then

$$\begin{aligned} &a_1(x,y) + a_4(x,y) + a_5(x,y) = a_1 + a_4 + a_5 < \frac{1}{s} \leq 1, \\ &a_2(x,y) + a_5(x,y) = a_2 + a_5 < \frac{1}{s}, \\ &a_3(x,y) + a_4(x,y) = a_3 + a_4 < \frac{1}{s}, L_i < +\infty \ \ (i = 1, 2, 3, 4, 5, 6). \end{aligned}$$

Since $(a_3 - a_2)(a_5 - a_4) > \frac{2\delta}{s^3}$, and $sa_1 < 1$, we have

$$\begin{aligned} & sa_1\delta + s^3a_2a_5 + s^3a_3a_4 < \delta + s^3a_2a_5 + s^3a_3a_4 < s^3a_3a_5 + s^3a_2a_4 - \delta; \\ & sa_1\delta + sa_1 + s^3a_2a_5 + s^3a_3a_4 < sa_1 + s^3a_3a_5 + s^3a_2a_4 - \delta; \\ & s^2a_1(a_1 + a_2 + a_3 + sa_4 + sa_5) + s^3a_2a_5 + s^3a_3a_4 + s^2a_2a_3 + s^4a_4a_5 \\ < \ \ & 1 - sa_2 - sa_3 - s^2a_4 - s^2a_5 + s^3a_2a_4 + s^3a_3a_5 + s^2a_2a_3 + s^4a_4a_5, \end{aligned}$$

which implies that

$$s^2(a_1 + a_3 + sa_5)(a_1 + a_2 + sa_4) < (1 - sa_2 - s^2a_5)(1 - sa_3 - s^2a_4).$$

Hence, we can obtain that

$$L_1L_2 = \frac{a_1+a_3+sa_5}{1-a_2-sa_5}\cdot\frac{a_1+a_2+sa_4}{1-a_3-sa_4} < \frac{a_1+a_3+sa_5}{1-sa_2-s^2a_5}\cdot\frac{a_1+a_2+sa_4}{1-sa_3-s^2a_4} < \frac{1}{s^2}.$$

From Theorem 3.1, we complete the proof of corollary.

Corollary 3.3 Let (X,d) be a cone b- metric space with the constant $s\geq 1$. Suppose that mappings $F,G,H,T:X\to X$ satisfy following conditions: for all $x,y\in X$,

$$d(Fx,Gy)\leq a_1d(Hx,Ty)+a_2d(Hx,Fx)+a_3d(Ty,Gy)+a_4d(Hx,Gy)+a_5d(Ty,Fx),$$

where $a_i\geq 0$ $(i=1,2,3,4,5)$ are nonnegative real number which satisfy that

$$a_1+a_2+a_3+sa_4+sa_5=\frac{1}{s},$$

and $a_3<a_2, a_5<a_4$ or $a_3>a_2, a_5>a_4$. If $F(X)\subseteq T(X), G(X)\subseteq H(X)$, one of $F(X),G(X),H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F,H\}$ and $\{G,T\}$ are weakly compatible pairs, then F,G,H and T have an unique common fixed point.

Proof Since $a_3<a_2, a_5<a_4$ or $a_3>a_2, a_5>a_4$, then $a_2+a_3\neq 0$. We can choose δ such that $0<\delta<\min\{\frac{s^3}{2}(a_3-a_2)(a_5-a_4), s(a_2+a_3)\}$. By setting $a_1'=a_1+\frac{\delta}{s}, a_i'=a_i$ $(i=2,3,4,5)$, we have

$$d(Fx,Gy)\leq a_1'd(Hx,Ty)+a_2'd(Hx,Fx)+a_3'd(Ty,Gy)+a_4'd(Hx,Gy)+a_5'd(Ty,Fx)$$

for all $x,y\in X$. It is easy to see that

$$\begin{aligned}
&a_1'+a_4'+a_5'=a_1+\frac{\delta}{s}+a_4+a_5\leq a_1+\frac{\delta}{s}+sa_4+sa_5=\frac{\delta}{s}+\frac{1}{s}-a_2-a_3<\frac{1}{s};\\
&a_2'+a_5'=a_2+a_5\leq a_2+sa_5<a_1+a_2+a_3+sa_4+sa_5=\frac{1}{s};\\
&a_3'+a_4'=a_3+a_4\leq a_3+sa_4<a_1+a_2+a_3+sa_4+sa_5=\frac{1}{s};\\
&a_1'+a_2'+a_3'+sa_4'+sa_5'=a_1+\frac{\delta}{s}+a_2+a_3+sa_4+sa_5=\frac{1+\delta}{s};\\
&(a_3'-a_2')(a_5'-a_4')=(a_3-a_2)(a_5-a_4)>\frac{2\delta}{s^3}.
\end{aligned}$$

So we know that the conclusions are true by Corollary 3.1.

Remark If $s=1$, that is, (X,d) is a cone metric space, and take $H=T$ in Corollary 3.2 and Corollary 3.3, we obtain Corollary 1 and Corollary 2 in [8].

Corollary 3.4 Let (X,d) be a cone b-metric space with the constant $s\geq 1$. Suppose that mappings $F,G,H,T:X\to X$ satisfy following conditions: for all $x,y\in X$,

$$d(Fx,Gy)\leq a_1d(Hx,Ty)+a_2d(Hx,Fx)+a_3d(Ty,Gy)+a_4d(Hx,Gy)+a_5d(Ty,Fx),$$

where $a_i\geq 0$ $(i=1,2,3,4,5)$ are nonnegative real number which satisfy that

$$a_1+a_2+a_3+2s\max(a_4,a_5)<\frac{1}{s}$$

or

$$a_1 + 2\max(a_2, a_3) + sa_4 + sa_5) < \frac{1}{s}.$$

If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof We take $a_i(x, y) = a_i$ $(i = 1, 2, 3, 4, 5)$ in Theorem 3.1. If

$$a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s},$$

then

$$\begin{aligned}
&a_1(x, y) + a_4(x, y) + a_5(x, y) = a_1 + a_4 + a_5\\
&\le a_1 + sa_4 + sa_5 \le a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s} \le 1,\\
&a_2(x, y) + a_5(x, y) = a_2 + a_5 \le a_2 + sa_5 \le a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s},\\
&a_3(x, y) + a_4(x, y) = a_3 + a_4 \le a_3 + sa_4 \le a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s},\\
&L_i < +\infty \ \ (i = 1, 2, 3, 4, 5, 6).
\end{aligned}$$

Since

$$\begin{aligned}
a_1 + a_2 + a_3 + 2sa_5 \le a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s},\\
a_1 + a_2 + a_3 + 2sa_4 \le a_1 + a_2 + a_3 + 2s\max(a_4, a_5) < \frac{1}{s},
\end{aligned}$$

then

$$a_1 + a_3 + sa_5 < \frac{1}{s} - a_2 - sa_5 = \frac{1 - sa_2 - s^2a_5}{s} \le \frac{1 - a_2 - sa_5}{s};$$

and

$$a_1 + a_2 + sa_4 < \frac{1}{s} - a_3 - sa_4 = \frac{1 - sa_3 - s^2a_4}{s} \le \frac{1 - a_3 - sa_4}{s},$$

which implies that

$$L_1 = \frac{a_1 + a_3 + sa_5}{1 - a_2 - sa_5} < \frac{1}{s}; L_2 = \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4} < \frac{1}{s}.$$

So $L_1L_2 < \frac{1}{s^2}$.

If $a_1 + 2\max(a_2, a_3) + sa_4 + sa_5) < \frac{1}{s}$, also we have

$$\begin{aligned}
&a_1(x, y) + a_4(x, y) + a_5(x, y) < 1, a_2(x, y) + a_5(x, y) < \frac{1}{s}, a_3(x, y) + a_4(x, y) < \frac{1}{s},\\
&L_i < +\infty \ \ (i = 1, 2, 3, 4, 5, 6).
\end{aligned}$$

Since

$$\begin{aligned}
a_1 + 2a_2 + sa_4 + sa_5 \le a_1 + 2\max(a_2, a_3) + sa_4 + sa_5 < \frac{1}{s},\\
a_1 + 2a_3 + sa_4 + sa_5 \le a_1 + 2\max(a_2, a_3) + sa_4 + sa_5 < \frac{1}{s},
\end{aligned}$$

then

$$a_1 + a_2 + sa_4 < \frac{1}{s} - a_2 - sa_5 = \frac{1 - sa_2 - s^2 a_5}{s} \leq \frac{1 - a_2 - sa_5}{s};$$

and

$$a_1 + a_3 + sa_5 < \frac{1}{s} - a_3 - sa_4 = \frac{1 - sa_3 - s^2 a_4}{s} \leq \frac{1 - a_3 - sa_4}{s}.$$

It follows that

$$L_1 L_2 = \frac{a_1 + a_3 + sa_5}{1 - a_2 - sa_5} \cdot \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4} < \frac{1}{s^2}.$$

By Theorem 3.1, we complete the proof.

Corollary 3.5 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$d(Fx, Gy) \leq \alpha d(Hx, Ty) + \beta(d(Hx, Fx) + d(Ty, Gy)) + \gamma(d(Hx, Gy) + d(Ty, Fx)),$$

where α, β, γ are nonnegative real number which satisfy that $\alpha + 2\beta + 2s\gamma < \frac{1}{s}$. If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Let $a_1 = \alpha, a_2 = a_3 = \beta, a_4 = a_5 = \gamma$ in Corollary 3.4.

Corollary 3.6 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$d(Fx, Gy) \leq \alpha d(Hx, Ty) + \beta(d(Hx, Fx) + d(Ty, Gy)),$$

where α, β are nonnegative real number which satisfy that $\alpha + 2\beta < \frac{1}{s}$. If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Let $a_1 = \alpha, a_2 = a_3 = \beta, a_4 = a_5 = 0$ in Corollary 3.4.

Corollary 3.7 Let (X, d) be a cone b-metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$d(Fx, Gy) \leq \lambda d(Hx, Fx) + \kappa d(Ty, Gy)),$$

where λ, κ are nonnegative real number which satisfy that $\lambda + \kappa < \frac{1}{s}$. If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Let $a_1 = a_4 = a_5 = 0, a_2 = \lambda, a_3 = \kappa$ in Corollary 3.4.

Corollary 3.8 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, T : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$d(Fx, Gy) \leq \lambda d(Hx, Gy) + \kappa d(Ty, Fx)),$$

where λ, κ are nonnegative real number which satisfy that $\lambda+\kappa < \frac{1}{s^2}$. If $F(X) \subseteq T(X), G(X) \subseteq H(X)$, one of $F(X), G(X), H(X)$ and $T(X)$ is a complete subspace of X, and both $\{F, H\}$ and $\{G, T\}$ are weakly compatible pairs, then F, G, H and T have an unique common fixed point.

Proof Let $a_1 = a_2 = a_3 = 0, a_4 = \lambda, a_5 = \kappa$ in Corollary 3.4.

Remark Corollary 3.4 improves Theorem 3.8 in [18], if $s = 1$ in Corollaries 3.4–3.8, we obtain Theorem 2.1 and Corollaries 2.4–2.7 in [15].

Example 1 Let $X = \{1, 2, 3\}, E = R^2, P = \{(x, y)/x \geq 0, y \geq 0\}$ and $d : X \times X \to E$ be defined as follows:

$$d(1,1) = d(2,2) = d(3,3) = (0,0);$$
$$d(1,2) = d(2,1) = (2,2), d(1,3) = d(3,1) = (10,10), d(2,3) = d(3,2) = (3,3).$$

Then it is easy to see that (X, d) is a cone b-metric space with constant $s = 2$. Mappings $F, G, H, T : X \to X$ is defined by $G(x) = 1$ for all $x \in X$ and

$$F(x) = \begin{cases} 1, x \neq 2, \\ 2, x = 2; \end{cases} \quad H(x) = \begin{cases} 1, x = 1, \\ 3, x \neq 1; \end{cases} \quad T(x) = \begin{cases} 1, x = 1, \\ 2, x \neq 1. \end{cases}$$

Let

$$a_1 = \frac{1}{50}, a_2 = \frac{1}{50}, a_3 = \frac{1}{125}, a_4 = \frac{9}{40}, a_5 = \frac{1}{1000},$$

then

$$a_1 + a_2 + a_3 + sa_4 + sa_5 = \frac{1}{s},$$

and $a_2 > a_3, a_4 > a_5$. We obtain

$$\begin{aligned}
(2,2) &= d(2,1) = d(F2, G3) \\
&< \frac{1}{50}d(H2, T3) + \frac{1}{50}d(H2, F2) + \frac{1}{125}d(T3, G3) + \frac{9}{40}d(H2, G3) + \frac{1}{1000}d(T3, F2) \\
&= \frac{1}{50}d(3,2) + \frac{1}{50}d(3,2) + \frac{1}{125}d(2,1) + \frac{9}{40}d(3,1) + \frac{1}{1000}d(2,2) \\
&= \frac{1}{50}(3,3) + \frac{1}{50}(3,3) + \frac{1}{125}(2,2) + \frac{9}{40}(10,10) = \frac{1193}{1000}(2,2).
\end{aligned}$$

Similarly, we know that other five inequalities are also true.

$$(0,0) = d(F1, G2) < \frac{1}{50}d(H1, T2) + \frac{1}{50}d(H1, F1) + \frac{1}{125}d(T2, G2) + \frac{9}{40}d(H1, G2) + \frac{1}{1000}d(T2, F1),$$
$$(0,0) = d(F1, G3) < \frac{1}{50}d(H1, T3) + \frac{1}{50}d(H1, F1) + \frac{1}{125}d(T3, G3) + \frac{9}{40}d(H1, G3) + \frac{1}{1000}d(T3, F1),$$
$$(0,0) = d(F3, G1) < \frac{1}{50}d(H3, T1) + \frac{1}{50}d(H3, F3) + \frac{1}{125}d(T1, G1) + \frac{9}{40}d(H3, G1) + \frac{1}{1000}d(T1, F3),$$
$$(0,0) = d(F3, G2) < \frac{1}{50}d(H3, T2) + \frac{1}{50}d(H3, F3) + \frac{1}{125}d(T2, G2) + \frac{9}{40}d(H3, G2) + \frac{1}{1000}d(T2, F3),$$
$$(2,2) = d(F2, G1) < \frac{1}{50}d(H2, T1) + \frac{1}{50}d(H2, F2) + \frac{1}{125}d(T1, G1) + \frac{9}{40}d(H2, G1) + \frac{1}{1000}d(T1, F2).$$

Thus, the all conditions in Corollary 3.2 are satisfied, F, G, H, T have an unique common fixed point $x^* = 1$.

It needs to mention that, for any nonnegative real number α, β, γ satisfying

$$\alpha + 2\beta + 2s\gamma < \frac{1}{s},$$

we have

$$\begin{aligned}
&\alpha d(H2, T3) + \beta(d(H2, F2) + d(T3, G3)) + \gamma(d(H2, G3) + d(T3, F2)) \\
=&\alpha d(3, 2) + \beta(d(3, 2) + d(2, 1)) + \gamma(d(3, 1) + d(2, 2)) \\
=&\alpha(3, 3) + \beta((3, 3) + (2, 2)) + \gamma(10, 10) \\
=&(\frac{3}{2}\alpha + \frac{5}{2}\beta + 5\gamma)(2, 2) = \frac{3}{2}(\alpha + \frac{5}{3}\beta + \frac{5}{3}s\gamma)(2, 2) \\
<&\frac{3}{2}(\alpha + 2\beta + 2s\gamma)(2, 2) < \frac{3}{2} \cdot \frac{1}{2}(2, 2) < (2, 2) \\
=&d(F2, G3).
\end{aligned}$$

So, the conditions in Corollary 3.5 are not satisfied.

Theorem 3.3 Let (X, d) be a cone b- metric space with the constant $s \geq 1$. Suppose that mappings $F, G, H, J, T, V : X \to X$ satisfy following conditions: for all $x, y \in X$,

$$\begin{aligned}
d(Fx, Gy) \quad \leq \quad & a_1(x, y)d(JTx, HVy) + a_2(x, y)d(JTx, Fx) + a_3(x, y)d(HVy, Gy) \\
& + a_4(x, y)d(JTx, Gy) + a_5(x, y)d(HVy, Fx), \qquad (3.10)
\end{aligned}$$

where $a_i(x, y)$ $(i = 1, 2, 3, 4, 5)$ satisfy same conditions as in Theorem 3.1. If

$$FJ = JF, JT = TJ, GH = HG, HV = VH, F(X) \subseteq HV(X), G(X) \subseteq JT(X),$$

one of $F(X), G(X), JT(X)$ and $HV(X)$ is a complete subspace of X, and both $\{F, JT\}$ and $\{G, HV\}$ are weakly compatible pairs, then F, G, H, J, T and V have an unique common fixed point.

Proof From Theorem 3.1, we know that F, G, JT and HV have an unique common fixed point q, that is,

$$Fq = Gq = JTq = HVq = q.$$

Since $FJ = JF, JT = TJ$, we have

$$\begin{aligned}
d(Jq, q) =&d(JFq, Gq) = d(FJq, Gq) \\
\leq&a_1(Jq, q)d(JTJq, HVq) \\
&+ a_2(Jq, q)d(JTJq, FJq) + a_3(Jq, q)d(HVq, Gq) \\
&+ a_4(Jq, q)d(JTJq, Gq) + a_5(Jq, q)d(HVq, FJq) \\
=&a_1(Jq, q)d(Jq, q) + a_2(Jq, q)d(Jq, Jq) + a_3(Jq, q)d(q, q) \\
&+ a_4(Jq, q)d(Jq, q) + a_5(Jq, q)d(q, Jq) \\
=&(a_1(Jq, q) + a_4(Jq, q) + a_5(Jq, q))d(Jq, q),
\end{aligned}$$

which implies that $d(Jq, q) = \theta$. So $Jq = q$, and $Tq = TJq = JTq = q$.

Similarly, we can obtain that $Hq = Vq = q$. Therefore,q is the unique common fixed point of F, G, H, J, T and V.

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锥b-度量空间上映射的公共不动点定理

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摘要: 本文研究了锥b-度量空间上四个自映射的公共不动点问题. 利用序列逼近的方法, 获得了锥b-度量空间上四个自映射的一些公共不动点结果, 将锥度量空间中的几个相关结果推广到锥b-度量空间中, 并且给出了一个例子以支撑我们的结果.

关键词: 锥b-度量空间; 公共不动点; 映射; 弱相容

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