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On Twisted Atiyah-Singer Operators (I) *

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Abstract In this paper we show that the de Rham and the Signature operators on a Riemannian manifold are all isomorphic to some twisted Atiyah-Singer operators. Then the local index theorem and local Lefschetz fixed point formulas of these operators can be obtained from the corresponding theorems of twisted Atiyah-Singer operators.

Keywords Clifford module, Spin group, elliptic operator, twisted Atiyah-Singer operator, index theorem.

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1. Introduction

Let M be an oriented compact Riemannian manifold of even dimension $2n$. There are two natural elliptic operators-the de Rham and the Signature operators on M. Furthermore, there is an Atiyah-Singer operator on spinor bundle over M if M has a spin structure. As explained in Atiyah et al.[1], the classical elliptic operators are locally the twisted Atiyah-Singer operators.

Let G be a Lie group with a Hermitian representation on a complex vector space V and $\triangle$ be an irreducible left module over $\mathbf{C}l_{2n}$. Let $\triangle(M)\otimes E$ be a vector bundle over M with fibres $\triangle\otimes V$ and structure group $\mathrm{Spin}(2n)\times_{Z_2}G$. Let $D:\Gamma(\triangle^+(M)\otimes E)\to\Gamma(\triangle^-(M)\otimes E)$ be a Dirac operator. Locally D is a twisted Atiyah-Singer operator (see Lawson and Michelsohn[2], II §5). We also call D the twisted Atiyah-Singer operator. Note that, the spinor bundle $\triangle(M)$ and the vector bundle E can be defined globally if M is spin. The names of the classical elliptic operators used in this paper follows [2] and [3].

In this paper we show that there are two twisted Atiyah-Singer operators D_1 and D_2 defined naturally, such that the de Rham and the Signature operators are isomorphic to $D_1+D_2^*$ and D_1+D_2 respectively. Then the local index theorem and the local Lefschetz fixed point formulas of them can be obtained from the corresponding theorems of the twisted Atiyah-Singer operators. In a subsequent paper, we shall study the Dolbeault operators on almost complex manifolds.

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2. The twisted Atiyah-Singer operators

Let M be an oriented compact Riemannian manifold with dimension $2n$ and $\wedge_{\mathbf{c}}(M) = \wedge(M) \otimes \mathbf{C}$ be the exterior algebra of M. The de Rham operator D_H and the Signature operator D_S can be formed by the following two fundmental elliptic operators:

$$\overline{D}_1 : \Gamma(\wedge_{\mathbf{c}}^{+}(M) \cap \wedge_{\mathbf{c}}^{\mathrm{even}}(M)) \to \Gamma(\wedge_{\mathbf{c}}^{-}(M) \cap \wedge_{\mathbf{c}}^{\mathrm{odd}}(M)),$$
$$\overline{D}_2 : \Gamma(\wedge_{\mathbf{c}}^{+}(M) \cap \wedge_{\mathbf{c}}^{\mathrm{odd}}(M)) \to \Gamma(\wedge_{\mathbf{c}}^{-}(M) \cap \wedge_{\mathbf{c}}^{\mathrm{even}}(M)),$$

where $\overline{D}_1 = \overline{D}_2 = d + \delta$, and $D_H = \overline{D}_1 + \overline{D}_2^*$, $D_S = \overline{D}_1 + \overline{D}_2$ (see [1]).

As is well known, the complex Clifford algebra $\mathbf{C}l_{2n}$ decomposes into a direct sum of irreducible left modules over $\mathbf{C}l_{2n}$. Denote one of these modules by $\triangle$. The space $\triangle' = \{\bar{\xi}^t | \xi \in \triangle\}$ is an irreducible right module over $\mathbf{C}l_{2n}$.

Lemma 2.1 $\mathbf{C}l_{2n} = \triangle \cdot \triangle'$, *where* $\cdot$ *denote the Clifford product. Then there is an isomorphism of bimodules:*

$$\mathbf{C}l_{2n} \cong \triangle \otimes \triangle'.$$

Proof Let $e_1, \cdots, e_{2n}$ be an orthonormal basis of $\mathbf{R}^{2n}$ and $\alpha_i = \sqrt{-1}e_{2i-1}e_{2i}$, $g_i = \frac{1}{2}(e_{2i-1} - \sqrt{-1}e_{2i})$, $\bar{g}_i = \frac{1}{2}(e_{2i-1} + \sqrt{-1}e_{2i}), i = 1, \cdots, n$. Let the $\{\alpha_i\}$ act on $\mathbf{C}l_{2n}$ from the right and decompose $\mathbf{C}l_{2n}$ into 2^n simultaneous eigenspaces of this action. Let $\triangle(\varepsilon_1, \cdots, \varepsilon_{2n})$ be one of such space, $\triangle(\varepsilon_1, \cdots, \varepsilon_{2n})$ be eigenspace of α_i with eigenvalue ε_i for $1 \leq i \leq n$, $\varepsilon_i = 1$ or -1. Then $\triangle = \triangle(-1, \cdots, -1)$ is generated by $\bar{g}_1 \cdots \bar{g}_n$ as left ideal of $\mathbf{C}l_{2n}$. It is easy to see that $\triangle \cdot g_1 \cdots g_n \bar{g}_{i_1} \cdots \bar{g}_{i_k} = \triangle(\tau_1, \cdots, \tau_n)$ with $\tau_{i_1} = \cdots = \tau_{i_k} = -1, \tau_p = 1$ for $p \neq i_1, \cdots, i_k$. Thus we have proved that

$$\mathbf{C}l_{2n} = \oplus \triangle(\varepsilon_1, \cdots, \varepsilon_n) = \triangle \cdot \triangle'.$$

Since $\dim \triangle \cdot \dim \triangle' = \dim \mathbf{C}l_{2n}$, there is a natural isomorphism: $\mathbf{C}l_{2n} = \triangle \cdot \triangle' \to \triangle \otimes \triangle'$ and the decomposition is invariant under the right and the left action of the elements of $\mathbf{C}l_{2n}$. The lemma has been proved. □

Let $P_{SO}(M)$ be the principal bundle of positively oriented orthonormal coframes and $\mathbf{C}l(M) = P_{SO}(M) \times_{Ad} \mathbf{C}l_{2n}$ the associated Clifford bundle, where $Ad(g) = Ad(\tilde{g}), \tilde{g} \in \mathrm{Spin}(2n)$ is a lift of g. Let $\triangle = \triangle^+ \oplus \triangle^-$. Then we have an isomorphism:

$$\mathbf{C}l(M) \cong [\triangle^+(M) \oplus \triangle^-(M)] \otimes [\triangle'^+(M) \oplus \triangle'^-(M)],$$

where $\triangle(M) \otimes \triangle'(M) = P_{SO}(M) \times_{Ad} (\triangle \otimes \triangle')$. There is a canonical connection ∇ on $\triangle(M) = \triangle^+(M) \oplus \triangle^-(M)$ determined by the Riemannian metric on M if M has a spin structure. Denote ∇' the corresponding connection on $\triangle'(M)$. The connection $\nabla \otimes \nabla'$ is well defined on $\triangle^{\pm}(M) \otimes \triangle'^{\pm}(M)$ even if $\triangle^{\pm}(M)$ and $\triangle'^{\pm}(M)$ can not be defined globally. Then there are two operators

$$D_1 : \Gamma(\triangle^+(M) \otimes \triangle'^+(M)) \to \Gamma(\triangle^-(M) \otimes \triangle'^+(M)),$$
$$D_2 : \Gamma(\triangle^+(M) \otimes \triangle'^-(M)) \to \Gamma(\triangle^-(M) \otimes \triangle'^-(M)),$$
$$D_1 = D_2 = D = \sum e_i(\nabla_{E_i} \otimes 1 + 1 \otimes \nabla'_{E_i}),$$

where $E_1, \cdots, E_{2n}$ is an oriented orthonormal basis of TM and $e_1, \cdots, e_{2n} \in \mathbf{C}l(M)$ are the corresponding elements. D_1 and D_2 are the twisted Atiyah-Singer operators. We shall show that the operators $\overline{D}_1$ and $\overline{D}_2$ are isomorphic to the operators D_1 and D_2 respectively. Then the de Rham and the Signature operators are essentially the twisted Atiyah-Singer operators.

Let $\omega^1, \cdots, \omega^{2n}$ be dual basis of $E_1, \cdots, E_{2n}$ and $e_j^{\pm} = \omega^j \pm i(E_j) : \wedge_{\mathbf{c}}(M) \to \wedge_{\mathbf{c}}(M)$ be operators, $j = 1, \cdots, 2n$. They are subject to the relations:

$$e_i^+ e_j^+ + e_j^+ e_i^+ = 2\delta_{ij}, \quad e_i^- e_j^- + e_j^- e_i^- = -2\delta_{ij}, \quad e_i^+ e_j^- + e_j^- e_i^+ = 0.$$

Proposition 2.2 *Under the canonical isomorphism $\rho : \wedge_{\mathbf{c}}(M) \to \mathbf{C}l(M)$, we have*

$$\begin{aligned}
\wedge_{\mathbf{c}}^+(M) \cap \wedge_{\mathbf{c}}^{\text{even}}(M) &\cong \triangle^+(M) \otimes \triangle'^+(M), \\
\wedge_{\mathbf{c}}^+(M) \cap \wedge_{\mathbf{c}}^{\text{odd}}(M) &\cong \triangle^+(M) \otimes \triangle'^-(M), \\
\wedge_{\mathbf{c}}^-(M) \cap \wedge_{\mathbf{c}}^{\text{even}}(M) &\cong \triangle^-(M) \otimes \triangle'^-(M), \\
\wedge_{\mathbf{c}}^-(M) \cap \wedge_{\mathbf{c}}^{\text{odd}}(M) &\cong \triangle^-(M) \otimes \triangle'^+(M).
\end{aligned}$$

If M has a spin structure, the proposition has been observed by Gilkey[3].

Proof It is easy to verify that $\rho(e_i^- \cdot \xi) = e_i \rho(\xi), \rho(\xi \cdot e_i^-) = \rho(\xi) e_i$ hold for any form ξ. Let $\tau = (\sqrt{-1})^n e_1^- \cdots e_{2n}^-$ and $\star$ be the Hodge star operator on $\wedge_{\mathbf{c}}(M)$. Then $\tau\xi = (\sqrt{-1})^{n+p(p-1)} \star \xi$ if ξ is a p-form and $\tau \cdot \eta \cdot \overline{\tau}^t = -\eta$ for any 1-form η. Then we have

$$\begin{aligned}
\wedge_{\mathbf{c}}^{\pm}(M) &= \{\xi \in \wedge_{\mathbf{c}}(M) | \tau\xi = \pm\xi\}, \\
\wedge_{\mathbf{c}}^{\text{even}}(M) &= \{\eta \in \wedge_{\mathbf{c}}(M) | \tau\eta\overline{\tau}^t = \eta\}, \\
\wedge_{\mathbf{c}}^{\text{odd}}(M) &= \{\zeta \in \wedge_{\mathbf{c}}(M) | \tau\zeta\overline{\tau}^t = -\zeta\}.
\end{aligned}$$

Let $\omega_{\mathbf{c}} = (\sqrt{-1})^n e_1 \cdots e_{2n}$ be the volume element of $\mathbf{C}l(M)$, then we have

$$\rho(\tau\xi) = \omega_{\mathbf{c}} \cdot \rho(\xi), \rho(\xi \cdot \overline{\tau}^t) = \rho(\xi) \cdot \overline{\omega}_{\mathbf{c}}^t.$$

The space $\triangle^\alpha(M) \otimes \triangle'^\beta(M)$ can also be defined by

$$\triangle^\alpha(M) \otimes \triangle'^\beta(M) = \{\psi \in \triangle(M) \otimes \triangle'(M) | \omega_{\mathbf{c}}\psi = \alpha\psi, \psi\overline{\omega}_{\mathbf{c}}^t = \beta\psi\},$$

where $\alpha, \beta = \pm$. This proves the proposition. □

Theorem 2.3 *For any $\xi \in \Gamma(\wedge_{\mathbf{c}}(M))$, we have*

$$\rho((d + \delta)\xi) = D\rho(\xi).$$

Proof Also denote ∇ the Levi-Civita connection, $\nabla_{E_i} E_j = \sum \Gamma_{ij}^k E_k$. By Yu[4], we have

$$d + \delta = \sum e_i^- \{\check{E}_i + \frac{1}{4} \sum \Gamma_{ij}^k (e_j^- e_k^- - e_j^+ e_k^+)\},$$

where $\check{E}_i$ is defined by $\check{E}_i(f\omega^{i_1} \cdots \omega^{i_l}) = (E_i f)\omega^{i_1} \cdots \omega^{i_l}$.

For any $\xi \in \Gamma(\wedge_c(M)), \rho(\xi) = \sum \zeta_q \otimes \eta_q \in \Gamma(\triangle(M) \otimes \triangle'(M))$, from [2], p.129, we have

$$-\frac{1}{4}\sum \Gamma_{ij}^k e_j^+ e_k^+ \xi = \frac{1}{4}\xi \cdot (\sum \Gamma_{ij}^k e_j^- e_k^-)^t.$$

Hence

$$\begin{aligned}\rho((d+\delta)\xi) &= \sum e_i\{\breve{E}_i(\zeta_q \otimes \eta_q) + \frac{1}{4}\sum \Gamma_{ij}^k e_j e_k \zeta_q \otimes \eta_q\} + \sum e_i \zeta_q \otimes \eta_q \cdot (\frac{1}{4}\sum \Gamma_{ij}^k e_j e_k)^t \\ &= \sum (e_i \nabla_{E_i} \zeta_q) \otimes \eta_q + \sum e_i \zeta_q \otimes \nabla'_{E_i} \eta_q = D(\sum(\zeta_q \otimes \eta_q)).\end{aligned}$$

This proves the theorem. □

Corollary 2.4 *The elliptic operators $\overline{D}_1$ and $\overline{D}_2$ are isomorphic to the twisted Atiyah-Singer operators D_1 and D_2 respectively.*

Then the local index theorem of de Rham and Signature operators is an easy consequence of the twisted Atiyah-Singer operators. The indexes of D_1 and D_2 are

$$\mathrm{ind}(D_k) = \frac{1}{2}(\mathrm{Sign}(M) + (-1)^{k+1}\chi(M)), \quad k = 1, 2.$$

Let M be a spin manifold and $\triangle^{\pm}(M)$ be spinor spaces on M. If ζ and η are parallel spinors on M, then $\zeta \otimes \overline{\eta}^t$ can be looked as a differential form on M. Then we have

Proposition 2.5 *Let $k^{\pm}$ be the dimension of space of parallel half spinors on M, then*

1) $(k^+)^2 + (k^-)^2 \le \sum_p \dim H^{2p}(M, R), 2k^+k^- \le \sum_p \dim H^{2p+1}(M, R)$;

2) $k^+(k^+ + k^-) \le \dim(\ker(d+\delta|_{\Gamma(\wedge_c^+(M))})), k^-(k^+ + k^-) \le \dim(\ker(d+\delta|_{\Gamma(\wedge_c^-(M))}))$.

3. Lefschetz fixed point fomulas

In what follows we show that the local Lefschetz fixed point formulas for the de Rham and the Signature operators can be derived from the corresponding formulas for twisted Atiyah-Singer operators.

Let $D_E : \Gamma(\triangle^+(M) \otimes E) \to \Gamma(\triangle^-(M) \otimes E)$ be a twisted Atiyah-Singer operator and $f : M \to M$ be an orientation preserve isometry. Let $\{N_i\}$ be the set of connected components of fixed points of f. It is well known that each N_i is a geodesic submanifold of M with even dimension. Let $\nu(N_i)$ be the normal bundle on N_i in M with a naturally defined orientation. The following theorem can be proved as [5].

Theorem 3.1 *If the isometry f can be lifted to a linear map $\overline{f}$ on $\triangle^{\pm}(M) \otimes E$. Then the Lefschetz number of f is*

$$L(f) = \sum_i \int_{N_i} \frac{(\sqrt{-1})^{n+\dim N_i} \hat{A}(\frac{\Omega^{\top}}{2\pi}) tr[f_E \cdot \exp(-\frac{\overline{\Omega}}{2\pi})]}{Pf[2sh(\frac{\Theta}{2} + \frac{\Omega^{\perp}}{4\pi})]},$$

where $\Omega^{\top}$, $\Omega^{\perp}$ and $\overline{\Omega}$ are the curvature matrices of TN_i, $\nu(N_i)$ and $E|_{N_i}$ respectively, $\exp\Theta$ is the matrix of $f_*|_{\nu(N_i)}$, $\overline{f} = \exp(-\frac{1}{4}\sum \Theta_{ij} e_i e_j) \otimes f_E$.

For de Rham and Signature operators, the lift of f always exist. If

$$f_*(E_1, \cdots, E_{2n})^t = \exp(A_{ij})(E_1, \cdots, E_{2n})^t,$$

then (see [6])

$$f^* = \exp(-\frac{1}{4}\sum A_{ij}e_i^- e_j^-)\cdot\exp(\frac{1}{4}\sum A_{ij}e_i^+ e_j^+) : \wedge_{\mathbf{C}}(M) \to \wedge_{\mathbf{C}}(M).$$

Under the isomorphism $\rho : \wedge_{\mathbf{C}}(M) \to \mathbf{C}l_{2n} \cong \triangle(M)\otimes\triangle'(M)$, f^* corresponds to

$$\overline{f} = Ad[\exp(-\frac{1}{4}\sum A_{ij}e_ie_j)],$$

where $\exp(-\frac{1}{4}\sum A_{ij}e_ie_j)$ is a left of $\exp(A_{ij})$. Hence the Lefschetz fixed point formulas for de Rham and Signature operators can be derived from Theorem 3.1.

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扭化的 Atiyah-Singer 算子 (I)

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摘 要

本文证明黎曼流形上的 de Rham 以及 Signature 算子都同构于扭化的 Atiyah-Singer 算子. 这两类算子的局部指数定理和局部 Lefschetz 不动点公式都可以从扭化的 Atiyah-Singer 算子得到.