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On Compositions of Generalized Fractional Integrals*

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Abstracts

Compositions of generalized fractional integral operators involving Gauss hypergeometric function with power weights are studied. Composition formulas for such integrals which are the operators of the same type are obtained. In particular, compositions of two identical operators are given.

I. Introduction

It is well-known that composition formulas for Riemann-Liouville fractional integrals and derivatives

$$(I^a_{0+}\varphi)(x)=\begin{cases}\dfrac{1}{\Gamma(a)}\displaystyle\int_0^x\frac{\varphi(t)\,dt}{(x-t)^{1-a}}, & \mathrm{Re}a>0,\ 0<x<\infty,\ a\in C,\\ \left(\dfrac{d}{dx}\right)^{[\mathrm{Re}(-a)]+1}(I^{1-[\mathrm{Re}(-a)]}_{0+}\varphi)(x), & \mathrm{Re}a<0,\end{cases}\tag{1}$$

$$(I^a_{-}\varphi)(x)=\begin{cases}\dfrac{1}{\Gamma(a)}\displaystyle\int_x^\infty\frac{\varphi(t)\,dt}{(t-x)^{1-a}}, & \mathrm{Re}a>0,\ 0<x<\infty,\ a\in C,\\ \left(-\dfrac{d}{dx}\right)^{[\mathrm{Re}(-a)]+1}(I^{1-[\mathrm{Re}(-a)]}_{-}\varphi)(x), & \mathrm{Re}a<0,\end{cases}\tag{2}$$

with power weights, see [1], are of importance in the theory of fractional calculus and the theory of integral and differential equations. The latters were also led to the appearance of some papers, see [2]—[12], devoted to an investigation of the generalized fractional integral operators of the form

$${}_1I^c_{0+}(a,b)\varphi(x)=\int_0^x\frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_2F_1(a,b;c;1-\frac{x}{t})\varphi(t)\,dt,\ 0<x<\infty,\ \mathrm{Re}c>0,\tag{3}$$

$${}_2I^c_{0+}(a,b)\varphi(x)=\int_0^x\frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_2F_1(a,b;c;1-\frac{t}{x})\varphi(t)\,dt,\ 0<x<\infty,\ \mathrm{Re}c>0,\tag{4}$$

$${}_3I^c_{-}(a,b)\varphi(x)=\int_x^\infty\frac{(t-x)^{c-1}}{\Gamma(c)}\,{}_2F_1(a,b;c;1-\frac{x}{t})\varphi(t)\,dt,\ 0<x<\infty,\ \mathrm{Re}c>0,\tag{5}$$

$${}_4I^c_{-}(a,b)\varphi(x)=\int_x^\infty\frac{(t-x)^{c-1}}{\Gamma(c)}\,{}_2F_1(a,b;c;1-\frac{t}{x})\varphi(t)\,dt,\ 0<x<\infty,\ \mathrm{Re}c>0,\tag{6}$$

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with the Gauss hypergeometric function ${}_2F_1(a, b; c; z)$, in the kernels.

The main purpose of the present paper is to study compositions of the operators (3)—(6) with power weights and various parameters a, b, and c which led to the integrals of the same form. In section 2, we prove a general proposition (Theorem 3) that characterizes the fractional integrals (3)—(6), and power weights whose compositions are the operators of the same type in some special functional spaces. Here we essentially use the index laws for fractional integrals and derivatives (1)—(2) (Theorem 1) and the formulas represented the operators (3)—(6) by the integrals (1)—(2) (Theorem 2). The latter was made early by E. R. Love [2]—[3] and O. L. Marichev [1, § 10] for L_p-functions, see Remark 1. In section 3, we obtain the composition formulas for two identical operators of the form (3)—(6) in the same special functional space (Theorem 4). The formulas of such kind were given by M. Saigo [5], see also [11]—[12], for the operators

$$(I_{0+}^{\alpha,\beta,\eta} f)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}) f(t)\, dt, \ \mathrm{Re}\alpha > 0, \quad (7)$$

$$(I_{-}^{\alpha,\beta,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}) f(t)\, dt, \ \mathrm{Re}\alpha > 0, \quad (B)$$

for weighted L_p-functions, see Remark 3.

2. Main theorems

Let γ be a nonnegative real number, σ be a contour $\sigma = \{1/2 - i\infty, 1/2 + i\infty\}$. We denote by $m_{0,\gamma}^{-1}(L)$ and $L_2^{(0,\gamma)}$ the spaces of functions $f(x)$, $0 < x < \infty$, represented by

$$f(x) = \frac{1}{2\pi i} \int_\sigma f^*(s) x^{-s} ds, \quad f^*(s) = s^{-\gamma} F(s) \quad (9)$$

where $F(s) \in L(\sigma)$ and $F(s) \in L_2(\sigma)$ correspondingly. We denote by $X_{\gamma,\delta}$ the space

$$X_{\gamma,\delta} = \{ f(x) : f(x) = x^\delta g(x) \quad \text{where} \quad g(x) \in m_{0,\gamma}^{-1}(L) \ \text{or} \ L_2^{(0,\gamma)} \} \quad (10)$$

Theorems 36. 17—36. 18 in [1] give us the proposition about compositions of fractional integrals and derivatives (1)—(2).

Theorem 1 Let I_{0+}^{α} and I_{-}^{α} be the operators (1) and (2). Let $(\alpha_1, \cdots, \alpha_n)$, $(\bar{\alpha}_1, \cdots, \bar{\alpha}_n)$, $(\beta_1, \cdots, \beta_m)$, $(\bar{\beta}_1, \cdots, \bar{\beta}_m)$ be four sets of any complex numbers,

$$\gamma = \max\{0, \max \sum_{j=1}^{k} (\bar{\alpha}_{i_j} - \alpha_{i_j}), \ \max \sum_{j=1}^{l} (\bar{\beta}_{v_j} - \beta_{v_j})$$

$$(i_1, \cdots, i_k) \subset (1, \cdots, n), \ (v_1, \cdots, v_e) \subset (1, \cdots, m)\}, \quad (11)$$

$$\max_{1 \le i \le n} \mathrm{Re}\, \alpha_i - \frac{1}{2} < \delta < \max_{1 \le j \le m} \mathrm{Re}\, \beta_j - \frac{1}{2} \quad (12)$$

Then the operators $x^{\bar{\alpha}_i} I_{0+}^{\alpha_i - \bar{\alpha}_i} x^{-\alpha_i}$ $(i = 1, 2, \cdots, n)$ and $x^{\bar{\beta}_j} I_{-}^{\beta_j - \bar{\beta}_j} x^{-\beta_j}$ $(j = 1, 2, \cdots, m)$ are commutative in the space $X_{\gamma,\delta}$. Moreover if $(\bar{\alpha}_1, \cdots, \bar{\alpha}_n)$ and $(\bar{\beta}_1, \cdots, \bar{\beta}_m)$ are some rearrangement of $(\alpha_1, \cdots, \alpha_n)$ and $(\beta_1, \cdots, \beta_m)$ correspondingly then composition of the above operators is an identical operator

$$\prod_{i=1}^{n}(x^{\bar{\alpha}_i} I_{0+}^{\alpha_i-\bar{\alpha}_i} x^{-\alpha_i}) \prod_{j=1}^{m}(x^{\overline{\quad}} I_{-}^{\beta_j-\bar{\beta}_j} x^{-\beta_j}) f(x) = f(x), \quad f(x) \in X_{\gamma,\delta} . \tag{13}$$

Applying Parseval equation for Mellin transform (9), see [1], we obtain

$$x^{\bar{\alpha}} I_{0+}^{\alpha-\bar{\alpha}} x^{-\alpha} f(x) = \frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(1-\alpha-s+\delta)}{\Pi(1-\bar{\alpha}-s+\delta)} g^{*}(s) x^{\delta-s} ds,$$

$$f(x) \in X_{\gamma,\delta}, \quad \gamma = \max(0, \mathrm{Re}(\bar{\alpha}-\alpha)), \quad \delta > \mathrm{Re}\,\alpha - \frac{1}{2}, \tag{14}$$

$$x^{\bar{\beta}} I_{-}^{\beta-\bar{\beta}} x^{-\beta} f(x) = \frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(\bar{\beta}+s)}{\Gamma(\beta+s)} g^{*}(s) x^{\delta-s} ds,$$

$$f(x) \in X_{\gamma,\delta}, \quad \gamma = \max(0, \mathrm{Re}(\bar{\beta}-\beta)), \quad \delta < \mathrm{Re}\,\alpha - \frac{1}{2}, \tag{15}$$

$$x^{\bar{\alpha}_1}{}_1 I_{0+}^{\alpha_1+\alpha_2-\bar{\alpha}_1-\bar{\alpha}_2}(\alpha_1-\bar{\alpha}_2, \alpha_2-\bar{\alpha}_2) x^{\bar{\alpha}_2-\alpha_1-\alpha_2} f(x)$$

$$= x^{\bar{\alpha}_1+\bar{\alpha}_2-\alpha_1}{}_2 I_{0+}^{\alpha_1+\alpha_2-\bar{\alpha}_1-\bar{\alpha}_2}(\alpha_1-\bar{\alpha}_2, \alpha_1-\bar{\alpha}_1) x^{-\alpha_2} f(x)$$

$$= \frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(1-\alpha_1-s+\delta)\Gamma(1-\alpha_2-s+\delta)}{\Gamma(1-\bar{\alpha}_1-s+\delta)\Gamma(1-\bar{\alpha}_2-s+\delta)} g^{*}(s) x^{\delta-s} ds, \quad f(x) \in X_{\gamma,\delta}, \tag{16}$$

$$\gamma = \max(0, \mathrm{Re}(\bar{\alpha}_1-\alpha_1), \mathrm{Re}(\bar{\alpha}_2-\alpha_2)), \quad \delta > \max(\mathrm{Re}\,\alpha_1, \mathrm{Re}\,\alpha_2) - \frac{1}{2}; \tag{17}$$

$$x^{\bar{\beta}_1+\bar{\beta}_2-\beta_1}{}_3 I_{-}^{\beta_1+\beta_2-\bar{\beta}_1-\bar{\beta}_2}(\beta_1-\bar{\beta}_2, \beta_1-\bar{\beta}_1) x^{-\beta_2} f(x)$$

$$= x^{\bar{\beta}_1}{}_3 I_{-}^{\beta_1+\beta_2-\bar{\beta}_1-\bar{\beta}_2}(\beta_1-\bar{\beta}_2, \beta_2-\bar{\beta}_2) x^{\bar{\beta}_2-\beta_1-\beta_2} f(x)$$

$$= \frac{1}{2\pi i}\int_{\sigma} \frac{\Gamma(\bar{\beta}_1+s)\Gamma(\bar{\beta}_2+s)}{\Gamma(\beta_1+s)\Gamma(\beta_2+s)} g^{*}(s) x^{\delta-s} ds, \quad f(x) \in X_{\gamma,\delta}, \tag{18}$$

$$\gamma = \max(0, \mathrm{Re}(\bar{\beta}_1-\beta_1), \mathrm{Re}(\bar{\beta}_2-\beta_2)), \quad \delta < \max(\mathrm{Re}\,\beta_1, \mathrm{Re}\,\beta_2) - \frac{1}{2}. \tag{19}$$

From these, we deduce the following theorem

Theorem 2 Let $\alpha_1, \alpha_2, \bar{\alpha}_1, \bar{\alpha}_2, \beta_1, \beta_2, \bar{\beta}_1, \bar{\beta}_2$ be any complex numbers, I_{0+}^{α}, I_{-}^{α}, ${}_1I_{0+}^{c}(a, b)$, ${}_2I_{0+}^{c}(a, b)$, ${}_3I_{-}^{c}(a, b)$, ${}_4I_{-}^{c}(a, b)$ be the operators (1)—(6), $X_{\gamma,\delta}$ be the space (10). If γ and δ satisfy the conditions (17) then

$$x^{\bar{\alpha}_1}{}_1 I_{0+}^{\alpha_1+\alpha_2-\bar{\alpha}_1-\bar{\alpha}_2}(\alpha_1-\bar{\alpha}_2, \alpha_2-\bar{\alpha}_2) x^{\bar{\alpha}_2-\alpha_1-\alpha_2} f(x)$$

$$= x^{\bar{\alpha}_1+\bar{\alpha}_2-\alpha_1}{}_2 I_{0+}^{\alpha_1+\alpha_2-\bar{\alpha}_1-\bar{\alpha}_2}(\alpha_1-\bar{\alpha}_2, \alpha_1-\bar{\alpha}_1) x^{-\alpha_2} f(x)$$

$$= (x^{\bar{\alpha}_1} I_{0+}^{\alpha_1-\alpha_2} x^{-\alpha_1})(x^{\bar{\alpha}_2} I_{0+}^{\alpha_2-\bar{\alpha}_2} x^{-\alpha_2}) f(x), \tag{20}$$

if γ and δ satisfy the conditions (19) then

$$= x^{\bar{\beta}_1}{}_3 I_{-}^{\beta_1+\beta_2-\bar{\beta}_1-\bar{\beta}_2}(\beta_1-\bar{\beta}_2, \beta_2-\bar{\beta}_2) x^{\bar{\beta}_2-\beta_1-\beta_2} f(x)$$

$$= x^{\bar{\beta}_1+\bar{\beta}_2-\beta_1}{}_4 I_{-}^{\beta_1+\beta_2-\bar{\beta}_1-\bar{\beta}_2}(\beta_1-\bar{\beta}_2, \beta_1-\bar{\beta}_1) x^{-\beta_2} f(x)$$

$$= (x^{\bar{\beta}_1} I_{-}^{\beta_1-\bar{\beta}_1} x^{-\beta_1})(x^{\bar{\beta}_2} I_{-}^{\beta_2-\bar{\beta}_2} x^{-\beta_2}) f(x). \tag{21}$$

Remark I When the operators (3) and (4) are taken on the finite segment $(0, d)$ and (5) and (6) on the halfaxis (d, ∞) $0 < d < \infty$, and $f(x) \in L_p(0, d)$ and $f(x) \in L_p(d, \infty)$ correspondingly, the equalities (20) and (21) have been obtained earlier by E. K. Love [2]—[3] for $p=1$ and O. I. Macrichev [1, 10] for $p>1$.

The following theorem characterizing various compositions of the operators

(3)—(6) which are the same type follow immediately from Theorem 1 and Theorem 2.

Theorem 3 Let $(\alpha_1, \cdots, \alpha_{2n})$, $(\beta_1, \cdots, \beta_{2m})$, $(\mu_1, \cdots, \mu_{2l})$, $(\nu_1, \cdots, \nu_{2k})$ be four sets of any complex numbers and $(\overline{\alpha}_1, \cdots, \overline{\alpha}_{2n})$, $(\overline{\beta}_1, \cdots, \overline{\beta}_{2m})$, $(\overline{\mu}_1, \cdots, \overline{\mu}_{2l})$, $(\overline{\nu}_1 \cdots, \overline{\nu}_{2k})$ be some of their rearrangements. Let $X_{\gamma,\delta}$ be the space (10) where

$$\gamma = \max[0, \max \mathrm{Re} \sum_{j=1}^{N} (\overline{\alpha}_{i_j} - \alpha_{i_j}),\ \max \mathrm{Re} \sum_{j=1}^{M} \overline{\beta}_{\overline{i}_j} - \beta_{\overline{i}_j}),$$

$$\max \mathrm{Re} \sum_{j=1}^{L} (\overline{\mu}_{i_j} - \mu_{i_j}),\ \max \mathrm{Re} \sum_{j=1}^{K} (\overline{\nu}_{\overline{s}_j} - \nu_{\overline{s}_j}),\ (i_1, \cdots, i_N) \subset (1, \cdots, 2n),$$

$$(\overline{i}_1, \cdots, \overline{i}_M) \subset (1, \cdots, 2m),\ (s_1, \cdots, s_L) \subset (1, \cdots, 2l),\ (\overline{s}_1, \cdots, \overline{s}_K) \subset (1, \cdots, 2\nu)] \quad (22)$$

$$\max_{\substack{1 \le i \le 2n \\ 1 \le j \le 2m}} (\mathrm{Re}\alpha_i, \mathrm{Re}\beta_j) - \frac{1}{2} < \delta < \max_{\substack{1 \le i \le 2l \\ 1 \le j \le 2K}} (\mathrm{Re}\mu_i, \mathrm{Re}\nu_j) \quad (23)$$

if $f(x) \in X_{\gamma,\delta}$ then

$$\begin{aligned}
&\prod_{i=1}^{n}{}' x^{\overline{\alpha}_i} {}_1I_{0+}^{\alpha_i + \alpha_{i+n} - \overline{\alpha}_i - \overline{\alpha}_{i+n}}(\alpha_i - \overline{\alpha}_{i+n}, \alpha_{i+n} - \overline{\alpha}_{i+n}) x^{\overline{\alpha}_{i+n} - \alpha_i - \alpha_{i+n}} \\
&\cdot \prod_{j=1}^{m}{}' x^{\overline{\beta}_j + \overline{\beta}_{j+m} - \beta_j} {}_2I_{0+}^{\beta_j + \beta_{j+m} - \overline{\beta}_j - \overline{\beta}_{j+m}}(\beta_j - \overline{\beta}_{j+m}, \beta_j - \overline{\beta}_j) x^{-\beta_{j+m}} \\
&\cdot \prod_{s=1}^{l}{}' x^{\overline{\mu}_s} {}_3I_{-}^{\mu_s + \mu_{s+l} - \overline{\mu}_s - \overline{\mu}_{s+l}}(\mu_s - \overline{\mu}_{s+l}, \mu_{s+l} - \overline{\mu}_{s+l}) x^{\overline{\mu}_{s+l} - \mu_s - \mu_{s+l}} \\
&\cdot \prod_{i=1}^{K}{}' x^{\overline{\nu}_\tau + \overline{\nu}_{\tau+k} - \nu_\tau} {}_3I_{-}^{\nu_\tau + \nu_{\tau+k} - \overline{\nu}_2 - \overline{\nu}_{\tau+k}}(\nu_\tau - \overline{\nu}_{\tau+k}, \nu_\tau - \overline{\nu}_\tau) x^{-\nu_\tau + k} f(x) \\
= &\prod_{i=1}^{n}{}'' x^{\alpha_i} {}_1I_{0+}^{\overline{\alpha}_i + \overline{\alpha}_{i+n} - \alpha_i - \alpha_{i+n}}(\overline{\alpha}_i - \alpha_{i+n}, \alpha_{i+a} - \overline{\alpha}_{i+n}) x^{\alpha_{i+n} - \overline{\alpha}_i - \overline{\alpha}_{i+n}} \\
&\cdot \prod_{j=1}^{m}{}'' x^{\beta_j + \beta_{j+m} - \overline{\beta}_j} {}_2I_{0+}^{\overline{\beta}_j + \overline{\beta}_{j+m} - \beta_j - \beta_{j+m}}(\overline{\beta}_j - \beta_{j+m}, \overline{\beta}_j - \beta_j) x^{-\overline{\beta}_{j+m}} \\
&\cdot \prod_{s=1}^{l}{}'' x^{\mu_s} {}_3I_{-}^{\overline{\mu}_s + \overline{\mu}_{s+l} - \mu_s - \mu_{s+l}}(\overline{\mu}_s - \mu_{s+l}, \overline{\mu}_{s+l} - \mu_{s+l}) x^{\mu_{s+l} - \overline{\mu}_s - \overline{\mu}_{s+l}} \\
&\cdot \prod_{\tau=1}^{k}{}'' x^{\nu_\tau + \nu_{\tau+k} - \overline{\nu}_2} {}_4I_{-}^{\overline{\nu}_\tau + \overline{\nu}_{\tau+k} - \nu_\tau - \nu_{\tau+k}}(\overline{\nu}_2 - \nu_{\tau+k}, \overline{\nu}_\tau - \nu_\tau) x^{-\overline{\nu}_{\tau+m}} f(x).
\end{aligned} \quad (24)$$

The signs $\prod_{i=1}^{n}{}'$ and $\prod_{i=1}^{n}{}''$ mean that the products are taken over such multipliers for which $\mathrm{Re}(\alpha_i + \alpha_{i+n} - \overline{\alpha}_i - \overline{\alpha}_{i+n}) > 0$ and $\mathrm{Re}(\alpha_i + \alpha_{i+n} - \overline{\alpha}_i - \overline{\alpha}_{i+n}) < 0$ respectively and the same for other products.

3. Compositions of two operators of identical type

We consider the particular case of Theorem 3 when there are in (24) only there operators of identical type. The corollary of Theorem 3 is the following theorem about the composition formulas for the operators (3)—(6).

Theorem 4 Let α_i $(i = 1, \cdots, 6)$ be any set of complex numbers and β_i $(i = 1, \cdots, 6)$ be some of their rearrangements such that $\mathrm{Re}(\alpha_1 + \alpha_2 - \beta_1 - \beta_2) > 0$, $\mathrm{Re}(\alpha_3 +$

$\alpha_4-\beta_3-\beta_4)>0$. Let $X_{\gamma,\delta}$ be the space (10) where $\delta\in R^1$ and

$$\gamma=\max[0,\max\sum_{j=1}^{k}(\beta_{i_j}-\alpha_{i_j}),\ (i_1,\cdots,i_k)\subset\{1,2,3,4,5,6\}] \tag{25}$$

If $f(x)\in X_{\gamma,\delta}$, $\delta>\max\limits_{1\le i\le 6}\mathrm{Re}\alpha_i-\frac{1}{2}$, then

$$x^{\beta_1}{}_1I_{0+}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_2-\beta_2)x^{\beta_2+\beta_3-\alpha_1-\alpha_2}{}_1I_{0+}^{\alpha_2+\alpha_3-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_4-\beta_4)$$
$$\cdot x^{\beta_4-\alpha_3-\alpha_4}f(x)=x^{\alpha_5}{}_1I_{0+}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_6-\alpha_6)x^{\alpha_6-\beta_5-\beta_6}f(x) \tag{26}$$
$$x^{\beta_1+\beta_2-\alpha_1}{}_2I_{0+}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_1-\beta_1)x^{\beta_3+\beta_4-\alpha_2-\alpha_3}{}_2I_{0+}^{\alpha_3+\alpha_4-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_3-\beta_3)$$
$$\cdot x^{-\alpha_4}f(x)=x^{\alpha_5+\alpha_6-\beta_5}{}_2I_{0+}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_5-\alpha_5)x^{-\beta_6}f(x) \tag{27}$$

If $f(x)\in X_{\gamma,\delta}$, $\delta<\max\limits_{1\le i\le 6}\mathrm{Re}\alpha_i-\frac{1}{2}$, then

$$x^{\beta_1}{}_3I_{-}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_2-\beta_2)x^{\beta_2+\beta_3-\alpha_1-\alpha_2}{}_3I_{-}^{\alpha_3+\alpha_4-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_4-\beta_4)$$
$$\cdot x^{\beta_4-\alpha_3-\alpha_4}f(x)=x^{\alpha_5}{}_3I_{-}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_6-\alpha_6)x^{\alpha_6-\beta_5-\beta_6}f(x), \tag{28}$$
$$x^{\beta_1+\beta_2-\alpha_1}{}_4I_{-}^{\alpha_1+\alpha_2-\beta_1-\beta_2}(\alpha_1-\beta_2,\alpha_1-\beta_1)x^{\beta_3+\beta_4-\alpha_2-\alpha_3}{}_4I_{-}^{\alpha_3+\alpha_4-\beta_3-\beta_4}(\alpha_3-\beta_4,\alpha_3-\beta_3)$$
$$\cdot x^{-\alpha_4}f(x)=x^{\alpha_5+\alpha_6-\beta_5}{}_4I_{-}^{\beta_5+\beta_6-\alpha_5-\alpha_6}(\beta_5-\alpha_6,\beta_5-\alpha_5)x^{-\beta_6}f(x). \tag{29}$$

According to the conditions of Theorem 4.

$$(\alpha_1+\alpha_2-\beta_1-\beta_2)+(\alpha_3+\alpha_4-\beta_3-\beta_4)=\beta_5+\beta_6-\alpha_5-\alpha_6. \tag{30}$$

Therefore the equalities (26)—(29) are some analogues of semigroup properties

$$I_{0+}^{\alpha}I_{0+}^{\beta}f=I_{0+}^{\alpha+\beta}f,\quad I_{-}^{\alpha}I_{-}^{\beta}f=I_{-}^{\alpha+\beta}f \tag{31}$$

for fractional operators (1) and (2). We consider some special cases of these formulas for the operators ${}_1I_{0+}$ and ${}_2I_{0+}$.

1. $\alpha_1=\beta_5$, $\alpha_2=\beta_3$, $\alpha_3=\beta_2$, $\alpha_4=\beta_6$, $\alpha_5=\beta_1$, $\alpha_6=\beta_4$.

We write

$$a=\alpha_1-\beta_2,\ b=\alpha_2-\beta_2,\ c=\alpha_1+\alpha_2-\beta_1-\beta_2,\ A=\beta_5-\alpha_6,$$
$$\bar a=\alpha_3-\beta_4,\ \bar b=\alpha_4-\beta_4,\ \bar c=\alpha_3+\alpha_4-\beta_3-\beta_4,\ B=\beta_6-\alpha_6. \tag{32}$$

One may check that $A=a+\bar a$, $B=\bar b=\bar c+b$ in this case. Therefore (26) and (27) reduce to

$${}_1I_{0+}^{c}(a,b)x^{-a}{}_1I_{0+}^{\bar c}(\bar a,\bar c+b)x^{a}f(x)={}_1I_{0+}^{c+\bar c}(a+\bar a,c+\bar c)f(x), \tag{33}$$
$$x^{\bar a}{}_1I_{0+}^{c}(a,c-b)x^{-\bar a}{}_2I_{0+}^{\bar c}(\bar a,-b)f(x)={}_2I_{0+}^{c+\bar c}(a+\bar a,c-b)f(x). \tag{34}$$

2. $\alpha_1=\beta_4$, $\alpha_2=\beta_3$, $\alpha_3=\beta_5$, $\alpha_4=\beta_6$, $\alpha_5=\beta_1$, $\alpha_6=\beta_2$,

$${}_1I_{0+}^{c}(a,b)x^{-a}{}_1I_{0+}^{\bar c}(\bar a,\bar c-a-\bar a+b)x^{a}f(x)={}_1I_{0+}^{c+\bar c}(a+\bar a,b+\bar c-\bar a)f(x), \tag{35}$$
$$x^{\bar a}{}_2I_{0+}^{c}(a,c-b)x^{-\bar a}{}_2I_{0+}^{\bar c}(\bar a,a+\bar a-b)f(x)={}_2I_{0+}^{c+\bar c}(a+\bar a,c+\bar a-b)f(x). \tag{36}$$

3. $\alpha_1=\beta_6$, $\alpha_2=\beta_3$, $\alpha_3=\alpha_4=\beta_5=\beta_6$, $\alpha_5=\beta_1$, $\alpha_6=\beta_4$,

$${}_1I_{0+}^{c}(a,b)x^{-a}{}_1I_{0+}^{\bar c}(b+\bar c,b+\bar c)x^{a}f(x)={}_1I_{0+}^{c+\bar c}(b+\bar c,a+b+\bar c)f(x), \tag{37}$$
$$x^{-a+b+\bar c}{}_2I_{0+}^{c}(a,c-b)x^{-b-\bar c}{}_2I_{0+}^{\bar c}(b+c,-b)x^{a}f(x)$$
$$={}_2I_{0+}^{c+\bar c}(b+\bar c,c-a-b)f(x). \tag{38}$$

4 . $a_1=\beta_4$, $a_2=\beta_3$, $a_3=\beta_2$, $a_4=\beta_6$, $a_5=\beta_1$, $a_6=\beta_5$,

$$ {}_1I_{0+}^{c}(a,b)x^{-a}{}_1I_{0+}^{\bar c}(-a,b+\bar c)x^{a}f(x)=I_{0+}^{c+\bar c}f(x), \tag{39}$$

$$ x^{-a}{}_2I_{0+}^{c}(a,c-b)x^{a}{}_2I_{0+}^{\bar c}(-a,-b)f(x)=I_{0+}^{c+\bar c}f(x). \tag{40}$$

5 . $a_1=\beta_2$, $a_2=\beta_3$, $a_3=\beta_5$, $a_4=\beta_6$, $a_5=\beta_1$, $a_6=\beta_5$,

$$ I_{0+}^{c}{}_1I_{0+}^{\bar c}(\bar a,\bar b)f(x)={}_1I_{0+}^{c+\bar c}(\bar a,\bar b)f(x), \tag{41}$$

$$ x^{\bar a}I_{0+}^{c}x^{-\bar a}{}_2I_{0+}^{\bar c}(\bar a,\bar c-\bar b)f(x)={}_2I_{0+}^{c+\bar c}(\bar a,c+\bar c-\bar b)f(x). \tag{42}$$

6 . $a_1=\beta_1$, $a_2=\beta_4$, $a_3=\beta_5$, $a_4=\beta_6$, $a_5=\beta_2$, $a_6=\beta_3$,

$$ I_{0+}^{c}x^{\bar a+\bar b-\bar c}{}_1I_{0+}^{\bar c}(\bar a,\bar b)x^{\bar c-\bar a-\bar b}f(x)={}_1I_{0+}^{c+\bar c}(\bar c-\bar a,\bar c-\bar b)f(x), \tag{43}$$

$$ x^{\bar a}I_{0+}^{c}x^{\bar b-\bar c}{}_2I_{0+}^{\bar c}(\bar a,\bar c-\bar b)x^{\bar c-\bar a-\bar b}f(x)={}_2I_{0+}^{c+\bar c}(\bar c-\bar a,c+\bar b)f(x). \tag{44}$$

7 . $a_1=\beta_5$, $a_2=\beta_3$, $a_3=\beta_4$, $a_4=\beta_6$, $a_5=\beta_1$, $a_6=\beta_3$,

$$ {}_1I_{0+}^{c}(a,b)x^{-a}I_{0+}^{\bar c}x^{a}f(x)={}_1I_{0+}^{c+\bar c}(a,b+\bar c)f(x), \tag{45}$$

$$ {}_2I_{0+}^{c}(a,c-b)I_{0+}^{\bar c}f(x)={}_2I_{0+}^{c+\bar c}(a,c-b)f(x). \tag{46}$$

8 . $a_1=\beta_1$, $a_2=\beta_5$, $a_3=\beta_3$, $a_4=\beta_6$, $a_5=\beta_2$, $a_6=\beta_4$,

$$ I_{0+}^{c}x^{-d}I_{0+}^{\bar c}x^{d}f(x)={}_1I_{0+}^{c+\bar c}(\bar c,d)f(x)=x^{-\bar c}{}_2I_{0+}^{c+\bar c}(\bar c,c+\bar c-d)x^{\bar c}f(x). \tag{47}$$

9 . $a_1=\beta_5$, $a_2=\beta_3$, $a_3=\beta_2$, $a_4=\beta_4$, $a_5=\beta_1$, $a_6=\beta_6$,

$$ {}_1I_{0+}^{c}(a,-\bar c)x^{-a}I_{0+}^{\bar c}x^{a}f(x)=I_{0+}^{c+\bar c}f(x), \tag{48}$$

$$ x^{-a}{}_2I_{0+}^{c}(a,c+\bar c)I_{0+}^{\bar c}x^{a}f(x)=I_{0+}^{c+\bar c}f(x). \tag{49}$$

10. $a_1=\beta_1$, $a_2=\beta_3$, $a_3=\beta_4$, $a_4=\beta_6$, $a_5=\beta_2$, $a_6=\beta_5$,

$$ I_{0+}^{c}I_{0+}^{\bar c}f(x)=I_{0+}^{c+\bar c}f(x). \tag{50}$$

Remark 2 The equalities (28) and (29) are obtained from (26) and (27) by taking ${}_3I_-$ and ${}_4I_-$ instead of ${}_1I_{0+}$ and ${}_2I_{0+}$. Therefore the formulas (33)—(50) are true with ${}_3I_-$, ${}_4I_-$ and I_- instead of ${}_1I_{0+}$, ${}_2I_{0+}$ and I_{0+}.

Remark 3 The connection between the operators (4) and (7) is given by

$$ {}_2I_{0+}^{c}(a,b)f(x)=x^{a}I_{0+}^{c,\,a-c,\,-b}f(x). \tag{51}$$

Therefore (34) and (36) can be rewritten in the form

$$ I_{0+}^{c,\,a-c,\,b-c}I_{0+}^{\bar c,\,\bar a-\bar c,\,b}f(x)=I_{0+}^{c+\bar c,\,a+\bar a-c-\bar c,\,b-c}f(x), \tag{52}$$

$$ I_{0+}^{c,\,a-c,\,b-c}I_{0+}^{\bar c,\,\bar a-\bar c,\,b-a-\bar a}f(x)=I_{0+}^{c+\bar c,\,a+\bar a-c-\bar c,\,\bar a-b-\bar c}f(x). \tag{53}$$

Using Remark 2 and connection

$$ {}_3I_-^{c}(a,b)f(x)=I_-^{c,\,a-c,\,-b}f(x)$$

between the operators (5) and (8) one may obtain formulas (52) and (53) for the operator $I_-^{\alpha,\beta,\gamma}$. The latter such as (52)—(53) coincide (after suitable change of parameters) with the ones given by M. Saigo[5] for the function $f(x)\in L_p((0,\infty),x^{\gamma})$, $1\le p<\infty$.

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