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DYNAMIC BEHAVIOURS OF AN AUTONOMOUS STAGE-STRUCTURED COMPETITIVE SYSTEMS WITH TOXIC EFFECT*†

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Abstract

An autonomous stage-structured competitive systems with toxic effect is investigated in this paper. Sufficient conditions which guarantee the global attractivity of the system and the extinction of the partial species are obtained, respectively. Our results supplement and compliment one of the main results of Liu and Li [Global stability analysis of a nonautonomous stage-structured competitive system with toxic effect and double maturation delays, Abstract and Applied Analysis, Volume 2014, Article ID 689573, 15 pages].

Keywords global attractivity; extinction; delay; toxic substance

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1 Introduction

Throughout this paper, we set:

$$f^M = \max_{t\in[0,\omega]} |f(t)|, \quad f^L = \min_{t\in[0,\omega]} |f(t)|,$$

where $f(t)$ is a ω-periodic continuous function.

It is well known that the effect of toxins on ecological systems is an important issue from mathematical and experimental points of view [1, 2]. In [3], Maynard Smith incorporated the effects of toxic substances in a two-species Lotka-Volterra competitive system by considering that each species produces a substance toxic to the other only when the other is present. The model takes the following form

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$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[K_1 - a_1x_1(t) - b_1x_2(t) - c_1x_1(t)x_2(t)], \\ \dot{x}_2(t) &= x_2(t)[K_2 - a_2x_2(t) - b_2x_1(t) - c_2x_1(t)x_2(t)], \end{aligned} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ represent the densities of two competing species at time t, respectively. K_1 and K_2 denote the birth rates of the first and second species, respectively. a_1 and a_2 are the rates of intraspecific competition term for the first and second species, respectively. b_1 and b_2 stand for the rates of interspecific competitions, respectively. c_1 and c_2 represent the toxic inhibition rates for the first species by the second species and vice versa.

However, the nonautonomous case is more realistic, according to system (1.1), Li and Chen [4] considered the following nonautonomous system of differential equations

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[b_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t) - d_1(t)x_1(t)x_2(t)], \\ \dot{x}_2(t) &= x_2(t)[b_2(t) - a_{21}(t)x_2(t) - a_{22}(t)x_1(t) - d_2(t)x_1(t)x_2(t)]. \end{aligned} \tag{1.2}$$

They showed that under some suitable conditions, one species will be driven to extinction while the other species stabilizes at a certain solution of a logistic equation. For more papers in this direction, one could refer to [5,7], [24,25] and the references cited therein.

Stage-structured models have been analyzed in many papers (see [8,12-20,23]). Recently, Li and Chen [8] proposed the following periodic competitive stage-structured Lotka-Volterra model with the effects of toxic substances

$$\begin{aligned} \dot{x}_1(t) &= b_1(t-\tau_1)\exp\Big(-\int_{t-\tau_1}^{t} r_1(s)\mathrm{d}s\Big)x_1(t-\tau_1) - a_{11}(t)x_1^2(t) \\ &\quad -a_{12}(t)x_1(t)x_2(t) - d_1(t)x_1^2(t)x_2(t), \\ \dot{y}_1(t) &= b_1(t)x_1(t) - r_1(t)y_1(t) - b_1(t-\tau_1)\exp\Big(-\int_{t-\tau_1}^{t} r_1(s)\mathrm{d}s\Big)x_1(t-\tau_1), \\ \dot{x}_2(t) &= b_2(t-\tau_2)\exp\Big(-\int_{t-\tau_2}^{t} r_2(s)\mathrm{d}s\Big)x_2(t-\tau_2) - a_{21}(t)x_1(t)x_2(t) \\ &\quad -a_{22}(t)x_2^2(t) - d_2(t)x_1(t)x_2^2(t), \\ \dot{y}_2(t) &= b_2(t)x_2(t) - r_2(t)y_2(t) - b_2(t-\tau_2)\exp\Big(-\int_{t-\tau_2}^{t} r_2(s)\mathrm{d}s\Big)x_2(t-\tau_2), \end{aligned} \tag{1.3}$$

where $x_i(t)$ and $y_i(t)$ $(i = 1, 2)$ represent the densities of mature and immature species at time $t > 0$, respectively. $b_i(t)$, $a_{ij}(t)$, $r_i(t)$, $d_i(t)$ $(i, j = 1, 2)$ are all nonnegative continuous and ω-periodic functions. They obtained a set of sufficient conditions which ensure the extinction of the second species and the global attractivity of the first species.

On the other hand, corresponding to system (1.1), the formulation of the allelopathic interaction as $c_1x_1^2(t)x_2(t)$, is the most simplest way to fulfill the concept of allelopathic interaction in which one species release toxic substance to the other species, and this effect is zero during the absence of affected species. And Chattopadhyay [9] demonstrated that an inhibitory allelopathic term has stabilizing effect on the competitive coexistence of two competitive phytoplankton species. Recently, Bandyopadhyay [10] considered an allelopathic phytoplankton model where two phytoplankton species compete for the access to the common nutrient with modified allelopathic interaction term as suggested by Sole et al. [11]. This model is governed by the following nonlinear ordinary differential equations

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)[K_1 - a_1x_1(t) - b_1x_2(t) - cx_1(t)x_2^2(t)],\\ \dot{x}_2(t) &= x_2(t)[K_2 - a_2x_2(t) - b_2x_1(t)],\end{aligned}\tag{1.4}$$

where c denotes the rate of toxic inhibition by the first species and the second species release toxic substance within the surrounding aquatic environment.

Stimulated by the works of Li and Chen [8] and Bandyopadhyay [10], Liu and Li [12] proposed the following periodic stage-structured competitive systems with toxic effect and double maturation delays

$$\begin{aligned}\dot{x}_1(t) &= \alpha_1(t)x_2(t) - \gamma_1(t)x_1(t) - \alpha_1(t-\tau_1)\exp\Big(-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s\Big)x_2(t-\tau_1),\\ \dot{x}_2(t) &= \alpha_1(t-\tau_1)\exp\Big(-\int_{t-\tau_1}^{t}\gamma_1(s)\mathrm{d}s\Big)x_2(t-\tau_1) - \beta_1(t)x_2^2(t)\\ &\quad -c_1(t)x_2(t)y_2(t) - \rho(t)x_2^2(t)y_2^2(t),\\ \dot{y}_1(t) &= \alpha_2(t)y_2(t) - \gamma_2(t)y_1(t) - \alpha_2(t-\tau_2)\exp\Big(-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s\Big)y_2(t-\tau_2),\\ \dot{y}_2(t) &= \alpha_2(t-\tau_2)\exp\Big(-\int_{t-\tau_2}^{t}\gamma_2(s)\mathrm{d}s\Big)y_2(t-\tau_2) - \beta_2(t)y_2^2(t)\\ &\quad -c_2(t)x_2(t)y_2(t),\end{aligned}\tag{1.5}$$

where $x_1(t)$ and $x_2(t)$ represent the densities of mature and immature species 1 at time $t > 0$, respectively; $y_1(t)$ and $y_2(t)$ represent the densities of mature and immature species 2 at time $t > 0$, respectively. $\alpha_i(t)$, $\beta_i(t)$, $r_i(t)$ $(i, j = 1, 2)$ and $\rho(t)$ are all nonnegative continuous and ω-periodic functions.

Concerned with the persistent property of system (1.5), they obtained the following result:

Theorem A *Assume that*

$$\alpha_1^L\beta_2^L > c_1^M\alpha_2^M, \quad \alpha_2^L\beta_1^L > c_2^M\alpha_1^M \tag{H$_0$}$$

hold, then system (1.5) *is permanent.*

Now we focus our attention on the autonomous stage-structured competition system

$$\begin{aligned}
\dot{x}_1(t) &= \alpha_1 x_2(t) - \gamma_1 x_1(t) - \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(t-\tau_1),\\
\dot{x}_2(t) &= \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(t-\tau_1) - \beta_1 x_2^2(t) - c_1 x_2(t) y_2(t) - \rho x_2^2(t) y_2^2(t),\\
\dot{y}_1(t) &= \alpha_2 y_2(t) - \gamma_2 y_1(t) - \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2),\\
\dot{y}_2(t) &= \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 y_2^2(t) - c_2 x_2(t) y_2(t)
\end{aligned} \tag{1.6}$$

together with the following initial conditions

$$\begin{aligned}
x_i(t) &= \varphi_i(t) > 0, \quad t \in (-\tau, 0],\ i = 1, 2,\\
y_i(t) &= \psi_i(t) > 0, \quad t \in (-\tau, 0],\ i = 1, 2,
\end{aligned} \tag{1.7}$$

where $\tau = \max\{\tau_1, \tau_2\}$, $\alpha_i,\ \beta_i,\ \gamma_i\ (i = 1, 2)$ and ρ are all positive constants.

For the continuity of the initial conditions, it is required that

$$x_1(0) = \int_{-\tau_1}^{0} \alpha_1 \varphi_2(s) \mathrm{e}^{\gamma_1 s} \mathrm{d}s, \quad y_1(0) = \int_{-\tau_2}^{0} \alpha_2 \psi_2(s) \mathrm{e}^{\gamma_2 s} \mathrm{d}s. \tag{1.8}$$

As a direct corollary of Theorem A, for system (1.6), we have the following result.

Corollary A *Assume that*

$$\alpha_1\beta_2 > c_1\alpha_2, \quad \alpha_2\beta_1 > c_2\alpha_1 \tag{H_0'}$$

hold, then system (1.6) *is permanent.*

Condition (H_0') is independent of $\tau_i,\ i = 1, 2$, which seems very strange, since one may be expected by introducing the stage structure of the species, the stage structure could influence on the dynamic behaviours of the system. Now lets consider the following example.

Example 1.1 Consider the following system

$$\begin{aligned}
\dot{x}_1(t) &= 3x_2(t) - x_1(t) - 3\mathrm{e}^{-2} x_2(t-2),\\
\dot{x}_2(t) &= 3\mathrm{e}^{-2} x_2(t-2) - 2x_2^2(t) - x_2(t) y_2(t) - 2x_2^2(t) y_2^2(t),\\
\dot{y}_1(t) &= 2y_2(t) - y_1(t) - 2\mathrm{e}^{-1} y_2(t-1),\\
\dot{y}_2(t) &= 2\mathrm{e}^{-1} y_2(t-1) - y_2^2(t) - x_2(t) y_2(t).
\end{aligned} \tag{1.9}$$

In this case, corresponding to system (1.6), one has

$$\begin{aligned}
&\alpha_1(t) = 3,\ \ \gamma_1(t) = 1,\ \ \beta_1(t) = 2,\ \ c_1(t) = 1,\ \ \rho(t) = 2,\\
&\alpha_2(t) = 2,\ \ \gamma_2(t) = 1,\ \ \beta_2(t) = 1,\ \ c_2(t) = 1,\ \ \tau_1 = 2,\ \ \tau_2 = 1.
\end{aligned}$$

By simple computation, one can see that

$$\alpha_1\beta_2 = 3 > 2 = c_1\alpha_2, \quad \alpha_2\beta_1 = 4 > 3 = c_2\alpha_1.$$

Clearly, condition (H_0') holds, but numeric simulation (Figure 1) shows that in this case species 1 will be driven to extinction.

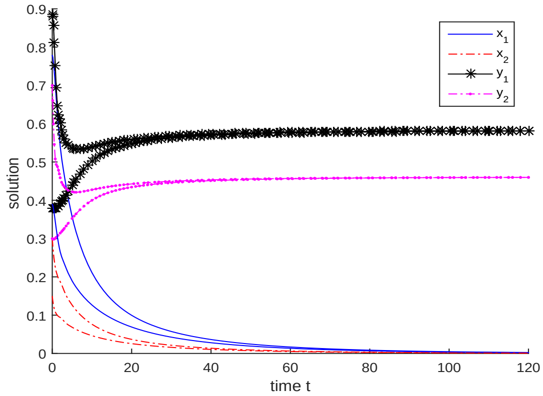

Figure 1: Dynamic behaviors of system (1.9) with initial values $(x_1(\theta), x_2(\theta), y_1(\theta), y_2(\theta))$ $= (0.7782, 0.3, 0.3793, 0.3)$ and $(0.3891,$ $0.15, 0.8850, 0.7)$, $\theta \in (-2, 0]$, respectively.

The above numeric simulation shows that although condition (H_0') holds, some of the species in system (1.6) maybe driven to extinction. Then two interesting issues are proposed:

1. If condition (H_0') is not enough to ensure the permanence of system (1.6), is it possible for us to find out some new sufficient conditions which ensure the permanence of the system? Further, based on the persistent result, is it possible to obtain sufficient conditions which ensure the global attractivity of the system?

2. Since numeric simulation shows that some of the species will be driven to extinction, while Liu and Li [12] did not investigate the extinction property of system (1.5), is it possible for us to obtain some sufficient conditions which ensure the extinction of system (1.6).

The organization of this paper is as follows. In Section 2, we introduce some useful lemmas. In Section 3, we study the global attractivity of system (1.6). In Section 4, we investigate the extinction property of the system. In Section 5, numerical simulations are presented to illustrate the feasibility of our main results.

2 Preliminaries

Now let us state several lemmas which will be useful in proving our main results.

Lemma 2.1[13] *Consider the following equation*

$$\dot{x}(t) = ax(t-\delta) - bx(t) - cx^2(t),$$
$$x(t) > 0, \quad -\delta \le t \le 0,$$

and assume that $a, b, c > 0$ *and* $\delta \ge 0$ *are constants, then:*

(i) *If* $a > b$, *then* $\lim\limits_{t\to+\infty} x(t) = \frac{a-b}{c}$;

(ii) *if* $a \le b$, *then* $\lim\limits_{t\to+\infty} x(t) = 0$.

Lemma 2.2[13] *Consider the following equation*

$$\dot{x}(t) = dx(t-\delta) - ex^2(t),$$
$$x(t) > 0, \quad -\sigma \le t \le 0,$$

and assume that $d, e > 0$ *and* $\sigma \ge 0$ *are constants, then*

$$\lim_{t\to+\infty} x(t) = \frac{d}{e}.$$

Lemma 2.3[21,22] (Fluctuation Lemma) *Let* $x(t)$ *be a bounded differentiable function on* (α, ∞). *Then there exist sequences* $\gamma_n \to \infty$, $\sigma_n \to \infty$ *such that*

(i) $\dot{x}(\gamma_n) \to 0$ *and* $x(\gamma_n) \to \limsup\limits_{t\to+\infty} x(t) = \overline{x}$ *as* $n \to \infty$;

(ii) $\dot{x}(\sigma_n) \to 0$ *and* $x(\sigma_n) \to \liminf\limits_{t\to+\infty} x(t) = \underline{x}$ *as* $n \to \infty$.

Lemma 2.4[12] *Solutions of system* (1.6) *with initial conditions* (1.7) *and* (1.8) *are positive for all* $t > 0$.

3 Global Attractivity

It follows from the first and the third equation of (1.6) that,

$$x_1(t) = \int_{t-\tau_1}^{t} \alpha_1 x_2(s) \mathrm{e}^{\gamma_1 s} \mathrm{d}s \cdot \mathrm{e}^{-\gamma_1 t}, \quad y_1(t) = \int_{t-\tau_2}^{t} \alpha_1 y_2(s) \mathrm{e}^{\gamma_2 s} \mathrm{d}s \cdot \mathrm{e}^{-\gamma_2 t}.$$

Hence we only need to study $x_2(t)$ and $y_2(t)$, which directly implies the properties of $x_1(t), y_1(t)$. Therefore, in this paper, we consider the subsystem of system (1.6) as follows

$$\begin{aligned} \dot{x}_2(t) &= \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(t-\tau_1) - \beta_1 x_2^2(t) - c_1 x_2(t) y_2(t) - \rho x_2^2(t) y_2^2(t), \\ \dot{y}_2(t) &= \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 y_2^2(t) - c_2 x_2(t) y_2(t), \end{aligned} \tag{3.1}$$

together with following initial conditions

$$x_2(t) = \varphi_2(t) > 0, \quad y_2(t) = \psi_2(t) > 0, \quad t \in (-\tau, 0],$$

where $\tau = \max\{\tau_1, \tau_2\}$. Before stating the main results of this section, we introduce a set of conditions

$$\frac{c_1}{\beta_2} < \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}} < \frac{\beta_1}{c_2}, \quad 0 < \rho < \frac{3\beta_2(\beta_1\beta_2 - c_1c_2)\mathrm{e}^{2\gamma_2\tau_2}}{\alpha_2^2}, \tag{H$_1$}$$

$$\frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}} \le \frac{c_1}{\beta_2}, \quad \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}} < \frac{\beta_1}{c_2}. \tag{H$_2$}$$

Lemma 3.1 *If* (H_1) *holds, then system* (3.1) *has a unique interior positive equilibrium* $E^*(x_2^*, y_2^*)$.

Proof From (18) and (22) in Liu and Li [12], for any $\varepsilon > 0$ enough small, there exists a T such that for all $t > T$,

$$x_2(t) \leq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon, \quad y_2(t) \leq \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2} + \varepsilon.$$

Hence, to investigate the positive equilibrium of system (3.1), it is enough to discuss the equilibrium on the region

$$D = \Big\{(x_2, y_2) |\, 0 \leq x_2(t) \leq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon,\ 0 \leq y_2(t) \leq \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2} + \varepsilon\Big\}.$$

The components of interior equilibrium point $E^*(x_2^*, y_2^*)$ are given by

$$\begin{aligned} &\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2^* - \beta_1 (x_2^*)^2 - c_1 x_2^* y_2^* - \rho (x_2^*)^2 (y_2^*)^2 = 0, \\ &\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2^* - \beta_2 (y_2^*)^2 - c_2 x_2^* y_2^* = 0, \end{aligned} \tag{3.2}$$

which is equivalent to

$$x_2^* = \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - \beta_2 y_2^*}{c_2},$$

and y_2^* is a positive real root of the equation

$$\beta_2 \rho z^3 - \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \rho z^2 + (\beta_1\beta_2 - c_1 c_2) z + \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} c_2 - \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \beta_1 = 0. \tag{3.3}$$

To end the proof of Lemma 3.1, it is enough to show that (3.3) admits a unique positive solution $z \in \big(0, \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2}\big)$. Let

$$f(z) = a z^3 + b z^2 + c z + d, \quad z \in \left(0, \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2}\right), \tag{3.4}$$

where $a = \beta_2 \rho > 0,\ b = -\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \rho < 0,\ c = \beta_1\beta_2 - c_1 c_2 > 0,\ d = \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} c_2 - \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} \beta_1 < 0$. Note that

$$f'(z) = 3az^2 + 2bz + c.$$

From the second inequality of condition (H_1)

$$\Delta = 4b^2 - 12ac = 4\alpha_2^2 \mathrm{e}^{-2\gamma_2\tau_2} \rho^2 - 12\beta_2\rho(\beta_1\beta_2 - c_1c_2) < 0.$$

Also, $f'(0) = c > 0$, hence $f'(z) > 0$ as $z \in \big(0, \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2}\big)$. That is, $f(z)$ is strictly increasing in the interval $\big(0, \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2}\big)$. Note that

$$f(0) = d < 0,$$

$$
\begin{aligned}
f\Big(\frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}\Big) &= \beta_2\rho\Big(\frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}\Big)^3 - \alpha_2 e^{-\gamma_2\tau_2}\rho\Big(\frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}\Big)^2 \\
&\quad +\big(\beta_1\beta_2 - c_1c_2\big)\frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2} + \alpha_1 e^{-\gamma_1\tau_1}c_2 - \alpha_2 e^{-\gamma_2\tau_2}\beta_1 \\
&= \alpha_2 e^{-\gamma_2\tau_2}\beta_1 - c_1c_2\frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2} + \alpha_1 e^{-\gamma_1\tau_1}c_2 - \alpha_2 e^{-\gamma_2\tau_2}\beta_1 \\
&> 0.
\end{aligned}
$$

Thus $f(z)$ has one and only one solution in the interval $\big(0, \frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2}\big)$, so we can obtain the existence and uniqueness of y_2^*, that is, system (3.1) has a unique interior positive equilibrium $E^*(x_2^*, y_2^*)$.

Lemma 3.2 *If* (H_1) *holds, then system* (1.6) *has a unique interior positive equilibrium* $E(x_1^*, x_2^*, y_1^*, y_2^*)$.

Theorem 3.1 *Let* $(x_2(t), y_2(t))^{\mathrm{T}}$ *be any solution of system* (3.1) *with initial conditions* (3.2). *Assume that the coefficients of system* (3.1) *satisfy condition* (H_1), *then the unique interior equilibrium* $E^*(x_2^*, y_2^*)$ *of system* (3.1) *is globally attractive, that is*

$$\lim_{t\to+\infty} x_2(t) = x_2^*, \qquad \lim_{t\to+\infty} y_2(t) = y_2^*.$$

Proof By the first equation of system (3.1), we have

$$\dot{x}_2(t) \le \alpha_1 e^{-\gamma_1\tau_1}x_2(t-\tau_1) - \beta_1 x_2^2(t). \tag{3.5}$$

From Lemma 2.2 and (3.5), there exists a $T_1 > 0$ such that for sufficiently small $\varepsilon > 0$ and $t \ge T_1$, we get

$$x_2(t) \le \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon \overset{\text{def}}{=} M_2^{(1)}. \tag{3.6}$$

Similarly, for the above $\varepsilon > 0$, from the second equation of system (3.1), it can be obtained that

$$\dot{y}_2(t) \le \alpha_2 e^{-\gamma_2\tau_2}y_2(t-\tau_2) - \beta_2 y_2^2(t), \quad \text{for } t \ge T_1. \tag{3.7}$$

By virtue of Lemma 2.2, there exists a $T_2 \ge T_1$ such that for the above $\varepsilon > 0$ and $t \ge T_2$, it yields

$$y_2(t) \le \frac{\alpha_2 e^{-\gamma_2\tau_2}}{\beta_2} + \varepsilon \overset{\text{def}}{=} M_4^{(1)}. \tag{3.8}$$

Furthermore, it follows from the first equation of system (3.1) that

$$\dot{x}_2(t) \ge \alpha_1 e^{-\gamma_1\tau_1}x_2(t-\tau_1) - c_1 M_4^{(1)}x_2(t) - \big(\beta_1 + \rho\big(M_4^{(1)}\big)^2\big)x_2^2(t). \tag{3.9}$$

From Lemma 2.1 and (3.9), for the above $\varepsilon > 0$, there exists a $T_3 > T_2$ such that for $t \ge T_3$, we get

$$x_2(t) \geq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(1)}}{\beta_1 + \rho\big(M_4^{(1)}\big)^2} - \varepsilon \overset{\text{def}}{=} m_2^{(1)} \tag{3.10}$$

provided that

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} > c_1 M_4^{(1)}. \tag{3.11}$$

It follows from the second equation of system (3.1) that

$$\dot{y}_2(t) \geq \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 x_2^2(t) - c_2 M_2^{(1)} y_2(t). \tag{3.12}$$

From Lemma 2.1 and (3.12), for the above $\varepsilon > 0$, there exists a $T_4 > T_3$ such that for $t \geq T_4$, we get

$$y_2(t) \geq \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(1)}}{\beta_2} - \varepsilon \overset{\text{def}}{=} m_4^{(1)} \tag{3.13}$$

provided that

$$\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} > c_2 M_2^{(1)}. \tag{3.14}$$

According to the first equation of system (3.1), we get that

$$\dot{x}_2(t) \leq \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(t-\tau_1) - c_1 m_4^{(1)} x_2(t) - (\beta_1 + \rho\big(m_4^{(1)}\big)^2) x_2^2(t). \tag{3.15}$$

From Lemma 2.1 and (3.15), for the above $\varepsilon > 0$, there exists a $T_5 > T_4$ such that for $t \geq T_5$, we get

$$x_2(t) \leq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 m_4^{(1)}}{\beta_1 + \rho\big(m_4^{(1)}\big)^2} + \frac{\varepsilon}{2} \overset{\text{def}}{=} M_2^{(2)} \tag{3.16}$$

provided that

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} > c_1 m_4^{(1)}. \tag{3.17}$$

According to the second equation of model system (3.1), we get that

$$\dot{y}_2(t) \leq \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 y_2^2(t) - c_1 m_2^{(1)} x_2(t). \tag{3.18}$$

From Lemma 2.1 and (3.18), for the above $\varepsilon > 0$, there exists a $T_6 > T_5$ such that for $t \geq T_6$, we get

$$y_2(t) \leq \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 m_2^{(1)}}{\beta_2} + \frac{\varepsilon}{2} \overset{\text{def}}{=} M_4^{(2)} \tag{3.19}$$

provided that

$$\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} > c_2 m_2^{(1)}. \tag{3.20}$$

Furthermore, it follows from the first equation of system (3.1) that

$$\dot{x}_2(t) \geq \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(t-\tau_1) - c_1 M_4^{(2)} x_2(t) - \big(\beta_1 + \rho\big(M_4^{(2)}\big)^2\big) x_2^2(t). \tag{3.21}$$

From Lemma 2.1 and (3.21), for the above $\varepsilon > 0$, there exists a $T_7 > T_6$ such that for $t \geq T_7$, we get

$$x_2(t) \geq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(2)}}{\beta_1 + \rho\big(M_4^{(2)}\big)^2} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_2^{(2)} \tag{3.22}$$

provided that

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} > c_1 M_4^{(2)}. \tag{3.23}$$

It follows from the second equation of system (3.1) that

$$\dot{y}_2(t) \geq \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 x_2^2(t) - c_2 M_2^{(2)} y_2(t). \tag{3.24}$$

From Lemma 2.1 and (3.24), for the above $\varepsilon > 0$, there exists a $T_8 > T_7$ such that for $t \geq T_8$, we get

$$y_2(t) \geq \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(2)}}{\beta_2} - \frac{\varepsilon}{2} \stackrel{\text{def}}{=} m_4^{(2)} \tag{3.25}$$

provided that

$$\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} > c_2 M_2^{(2)}. \tag{3.26}$$

It is easy to show that six inequalities (3.11), (3.14), (3.17), (3.20), (3.23) and (3.26) hold if the first inequality of (H_1) holds. Obviously

$$\begin{aligned}
M_2^{(2)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 m_4^{(1)}}{\beta_1 + \rho\big(m_4^{(1)}\big)^2} + \frac{\varepsilon}{2} < \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon = M_2^{(1)},\\
M_4^{(2)} &= \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 m_2^{(1)}}{\beta_2} + \frac{\varepsilon}{2} < \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{\beta_2} + \varepsilon = M_4^{(1)},\\
m_2^{(2)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(2)}}{\beta_1 + \rho\big(M_4^{(2)}\big)^2} - \frac{\varepsilon}{2} > \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(1)}}{\beta_1 + \rho\big(M_4^{(1)}\big)^2} - \varepsilon = m_2^{(1)},\\
m_4^{(2)} &= \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(2)}}{\beta_2} - \frac{\varepsilon}{2} > \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(1)}}{\beta_2} - \varepsilon = m_4^{(1)}.
\end{aligned} \tag{3.27}$$

Therefore

$$\begin{aligned}
&0 < m_2^{(1)} < m_2^{(2)} < x_2(t) < M_2^{(2)} < M_2^{(1)},\\
&0 < m_4^{(1)} < m_4^{(2)} < y_2(t) < M_4^{(2)} < M_4^{(1)}, \quad t \geq T_8.
\end{aligned} \tag{3.28}$$

Furthermore, four sequences will be obtained by repeating the discussion in this manner, which are given as follows

$$\begin{aligned}
M_2^{(n)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 m_4^{(n-1)}}{\beta_1 + \rho\big(m_4^{(n-1)}\big)^2} + \frac{\varepsilon}{n}, \quad M_4^{(n)} = \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 m_2^{(n-1)}}{\beta_2} + \frac{\varepsilon}{n},\\
m_2^{(n)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(n)}}{\beta_1 + \rho\big(M_4^{(n)}\big)^2} - \frac{\varepsilon}{n}, \quad m_4^{(n)} = \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(n)}}{\beta_2} - \frac{\varepsilon}{n}.
\end{aligned} \tag{3.29}$$

We claim that the sequences $\{M_i^{(n)}\}$ $(i=2,4)$ are strictly decreasing as n increases and the sequences $\{m_i^{(n)}\}$ $(i=2,4)$ are strictly increasing as n increases. We prove this claim. Firstly, from (3.28), we get

$$M_i^{(2)} < M_i^{(1)}, \quad m_i^{(2)} > m_i^{(1)}, \quad i=2,4.$$

Assume that our claim is true for n, so

$$M_i^{(n)} < M_i^{(n-1)}, \quad m_i^{(n)} > m_i^{(n-1)}, \quad i=2,4.$$

By a simple computation, we obtain that

$$\begin{aligned}
M_2^{(n+1)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 m_4^{(n)}}{\beta_1 + \rho\big(m_4^{(n)}\big)^2} + \frac{\varepsilon}{n+1} < \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 m_4^{(n-1)}}{\beta_1 + \rho\big(m_4^{(n-1)}\big)^2} + \frac{\varepsilon}{n} = M_2^{(n)},\\
M_4^{(n+1)} &= \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 m_2^{(n)}}{\beta_2} + \frac{\varepsilon}{n+1} < \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 m_2^{(n-1)}}{\beta_2} + \frac{\varepsilon}{n} = M_4^{(n)},\\
m_2^{(n+1)} &= \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(n+1)}}{\beta_1 + \rho\big(M_4^{(n+1)}\big)^2} - \frac{\varepsilon}{n+1} > \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1} - c_1 M_4^{(n)}}{\beta_1 + \rho\big(M_4^{(n)}\big)^2} - \frac{\varepsilon}{n} = m_2^{(n)},\\
m_4^{(n+1)} &= \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(n+1)}}{\beta_2} - \frac{\varepsilon}{n+1} > \frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 M_2^{(n)}}{\beta_2} - \frac{\varepsilon}{n} = m_4^{(n)}.
\end{aligned} \tag{3.30}$$

Therefore

$$\begin{aligned}
&0 < m_2^{(1)} < m_2^{(2)} < \cdots < m_2^{(n)} < x_2(t) < M_2^{(n)} < \cdots < M_2^{(2)} < M_2^{(1)},\\
&0 < m_4^{(1)} < m_4^{(2)} < \cdots < m_4^{(n)} < y_2(t) < M_4^{(n)} < \cdots < M_4^{(2)} < M_4^{(1)}, \quad t \geq T_{4n}.
\end{aligned} \tag{3.31}$$

Hence, the limits of $M_i^{(n)}, m_i^{(n)}, \ i=2,4, \ n=1,2,\cdots$ exist. Denote that

$$\lim_{n\to+\infty} M_2^{(n)} = \overline{x}_2, \quad \lim_{n\to+\infty} M_4^{(n)} = \overline{y}_2,$$
$$\lim_{n\to+\infty} m_2^{(n)} = \underline{x}_2, \quad \lim_{n\to+\infty} m_4^{(n)} = \underline{y}_2.$$

Letting $n \to +\infty$ in (3.29), we obtain

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}\overline{x}_2 - \beta_1(\overline{x}_2)^2 - c_1\overline{x}_2\underline{y}_2 - \rho(\overline{x}_2)^2(\underline{y}_2)^2 = 0 = \alpha_2\mathrm{e}^{-\gamma_2\tau_2}\underline{y}_2 - \beta_2(\underline{y}_2)^2 - c_2\overline{x}_2\underline{y}_2,$$

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}\underline{x}_2 - \beta_1(\underline{x}_2)^2 - c_1\underline{x}_2\overline{y}_2 - \rho(\underline{x}_2)^2(\overline{y}_2)^2 = 0 = \alpha_2\mathrm{e}^{-\gamma_2\tau_2}\overline{y}_2 - \beta_2(\overline{y}_2)^2 - c_2\underline{x}_2\overline{y}_2.$$

Note that $(\overline{x}_2, \underline{y}_2)$ and $(\underline{x}_2, \overline{y}_2)$ are positive solutions of (3.2), and

$$\overline{x}_2 < M_2^{(1)} = \frac{\alpha_1\mathrm{e}^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon, \quad \overline{y}_2 < M_4^{(1)} = \frac{\alpha_2\mathrm{e}^{-\gamma_2\tau_2}}{\beta_2} + \varepsilon.$$

By Lemma 3.1, (3.2) has a unique positive solution $E^*(x_2^*, y_2^*) \in D$. Hence, we conclude that

$$\overline{x}_2 = \underline{x}_2 = x_2^*, \quad \overline{y}_2 = \underline{y}_2 = y_2^*,$$

that is

$$\lim_{t\to+\infty} x_2(t) = x_2^*, \quad \lim_{t\to+\infty} y_2(t) = y_2^*.$$

Therefore, the unique interior equilibrium $E^*(x_2^*, y_2^*)$ of system (3.1) is globally attractive.

Corollary 3.1 *Let* $(x_1(t), x_2(t), y_1(t), y_2(t))^{\mathrm{T}}$ *be any solution of system* (1.6) *with initial conditions* (1.7). *Assume that the coefficients of system* (1.6) *satisfy inequality* (H_1), *then the unique interior equilibrium* $E(x_1^*, x_2^*, y_1^*, y_2^*)$ *of system* (1.6) *is globally attractive, that is*

$$\lim_{t\to+\infty} x_i(t) = x_i^*, \quad \lim_{t\to+\infty} y_i(t) = y_i^*, \quad i = 1, 2.$$

4 Extinction

Lemma 4.1 *Let* $(x_1(t), x_2(t), y_1(t), y_2(t))^{\mathrm{T}}$ *be any solution of system* (1.6) *with initial conditions* (1.7). *Assume that* (H_2) *holds, then there exists an* $\alpha > 0$ *such that* $y_2(t) \geq \alpha$ *for all* $t \geq 0$.

Proof It follows from Lemma 2.3 that $\limsup\limits_{t\to+\infty} x_2(t) \leq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1}$. Given $\varepsilon_1 = \frac{1}{2}\big(\frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{c_2} - \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1}\big) > 0$, there exists a $T \geq 0$ such that for $t \geq T$

$$x_2(t) \leq \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1} + \varepsilon_1 = \frac{1}{2}\Big(\frac{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}{c_2} + \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1}\Big).$$

So, for $t \geq T$

$$\begin{aligned} \dot{y}_2(t) &= \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \beta_2 y_2^2(t) - c_2 x_2(t) y_2(t) \\ &\geq \alpha_2 \mathrm{e}^{-\gamma_2\tau_2} y_2(t-\tau_2) - \frac{1}{2}\Big(\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} + c_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1}\Big) y_2(t) - \beta_2 y_2^2(t) \\ &\overset{\text{def}}{=} A y_2(t-\tau_2) - B y_2(t) - C y_2^2(t). \end{aligned}$$

Let $u(t)$ be a solution of the following equation

$$\dot{u}(t) = A u(t-\tau_2) - B u(t) - C u^2(t)$$

with $u(T+\tau_2) = y_2(T+\tau_2)$. It follows from condition (H_2) that

$$A - B = \frac{1}{2}\Big(\alpha_2 \mathrm{e}^{-\gamma_2\tau_2} - c_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\beta_1}\Big) > 0.$$

From Lemma 2.2

$$\lim_{t\to+\infty} u(t) = \frac{A-B}{C} \overset{\text{def}}{=} \alpha_1 > 0.$$

Therefore, we obtain

$$\underline{y}_2 = \liminf_{t\to+\infty} y_2(t) \geq \alpha_1.$$

Given $\varepsilon_2 = \alpha_1/2$, there exists a $T_1 \geq T$ such that

$$y_2(t) \geq \underline{y}_1 - \varepsilon_2 \geq \alpha_1 - \alpha_1/2 \geq \alpha_1/2, \quad t \geq T_1.$$

Let $\alpha_2 = \min\{y_2(t) : 0 \leq t \leq T_1\} > 0$ and $\alpha = \min\{\alpha_1/2, \alpha_2\} > 0$. It follows that $y_2(t) \geq \alpha > 0$ for all $t \geq 0$. This completes the proof of Lemma 4.1.

Now we show that (H_0) together with (H_2) could lead to the extinction of the first species, that is, in addition to (H_0), with some additional restriction, then the first species will be driven to extinction.

Theorem 4.1 *Let* $(x_1(t), x_2(t), y_1(t), y_2(t))^{\mathrm{T}}$ *be any solution of system* (1.6) *with initial conditions* (1.7). *Assume that* (H_0) *and* (H_2) *hold, then*

$$\lim_{t\to+\infty} x_1(t) = 0, \quad \lim_{t\to+\infty} x_2(t) = 0.$$

Proof By Lemma 2.4 we know that $x_2(t)$ and $y_2(t)$ are bounded and positive for all $t \geq 0$. Let $\overline{x}_2 = \limsup\limits_{t\to+\infty} x_2(t)$ and $\underline{y}_2 = \liminf\limits_{t\to+\infty} y_2(t)$. From Lemma 4.1 we know that $\underline{y}_2 \geq \alpha > 0$. Obviously $\overline{x}_2 \geq 0$. To prove $\lim\limits_{t\to+\infty} x_2(t) = 0$, it suffices to show that $\overline{x}_2 = 0$. In order to get a contradiction, we suppose that $\overline{x}_2 > 0$. According to the Fluctuation lemma, there exist sequences $\gamma_n \to +\infty$, $\sigma_n \to +\infty$ such that $\dot{x}_2(\gamma_n) \to 0$, $\dot{y}_2(\sigma_n) \to 0$, $x_2(\gamma_n) \to \overline{x}_2$ and $y_2(\sigma_n) \to \underline{y}_2$ as $n \to \infty$.

It follows from the second equation of system (1.6) that

$$\begin{aligned}\dot{x}_2(\gamma_n) &\leq \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} x_2(\gamma_n - \tau_1) - \beta_1 x_2^2(\gamma_n) - c_1 x_2(\gamma_n) y_2(\gamma_n)\\ &\leq \alpha_1 \mathrm{e}^{-\gamma_1\tau_1} \sup_{t\geq\gamma_n-\tau_1} x_2(t) - \beta_1 x_2^2(\gamma_n) - c_1 x_2(\gamma_n) \inf_{t\geq\gamma_n} y_2(t).\end{aligned}$$

By taking the limit of the above inequality as $n \to +\infty$, we obtain the inequality

$$0 \leq \alpha_1 \mathrm{e}^{-\gamma_1\tau_1}\overline{x}_2 - \beta_1 \overline{x}_2^2 - c_1 \overline{x}_2 \underline{y}_2.$$

That is

$$\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}\overline{x}_2 \geq \beta_1 \overline{x}_2^2 + c_1 \overline{x}_2 \underline{y}_2. \tag{4.1}$$

From the fourth equation of system (1.6), by a similar argument as above, we obtain

$$\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}\underline{y}_2 \leq \beta_2 \underline{y}_2^2 + c_2 \overline{x}_2 \underline{y}_2. \tag{4.2}$$

Multiplying (4.2) by $-\frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}\overline{x}_2$, we obtain

$$-\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}\overline{x}_2\underline{y}_2 \geq -\beta_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}\overline{x}_2\underline{y}_2^2 - c_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}\overline{x}_2^2\underline{y}_2. \tag{4.3}$$

Multiplying (4.1) by $\underline{y}_2$ and adding the corresponding inequality to (4.3), we obtain

$$0 \geq \Big(\beta_1 - c_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}\Big)\overline{x}_2^2\underline{y}_2 + \Big(c_1 - \beta_2 \frac{\alpha_1 \mathrm{e}^{-\gamma_1\tau_1}}{\alpha_2 \mathrm{e}^{-\gamma_2\tau_2}}\Big)\overline{x}_2\underline{y}_2^2.$$

That is

$$\Big(c_2\frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}} - \beta_1\Big)\overline{x}_2 \geq \Big(c_1 - \beta_2\frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}}\Big)\underline{y}_2. \tag{4.4}$$

From the first inequality in condition (H_2) and (4.4), we have

$$\Big(c_2\frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}} - \beta_1\Big)\overline{x}_2 \geq 0. \tag{4.5}$$

From the second inequality in condition (H_2), we have

$$c_2\frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}} - \beta_1 < 0. \tag{4.6}$$

This together with (4.5) leads to $\overline{x}_2 \leq 0$, which is a contradiction, then we obtain

$$\lim_{t\to+\infty} x_2(t) = 0. \tag{4.7}$$

Hence, for $0 < \varepsilon < \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\beta_1}$ sufficiently small, there exists a $T_1 > T$ such that $x_2(t) \leq \varepsilon$. Then we get

$$\begin{aligned} x_1(t) &= \int_{t-\tau_1}^{t} \alpha_1 x_2(s) e^{\gamma_1 s} ds \cdot e^{-\gamma_1 t} \leq \alpha_1\varepsilon \int_{t-\tau_1}^{t} \frac{1}{\gamma_1}\gamma_1 e^{\gamma_1 s} ds \cdot e^{-\gamma_1 t} \\ &= \frac{\alpha_1\varepsilon}{\gamma_1}\big(1 - e^{-\gamma_1\tau_1}\big), \quad t > T_1 + \tau_1. \end{aligned}$$

Setting $\varepsilon \to 0$, it follows that

$$\lim_{t\to+\infty} x_1(t) = 0. \tag{4.8}$$

This completes the proof of Theorem 4.1.

5 Examples

Example 5.1 Consider the following autonomous stage-structured competitive systems with toxic effect and double maturation delays

$$\begin{aligned} \dot{x}_1(t) &= 3x_2(t) - x_1(t) - 3e^{-2}x_2(t-2), \\ \dot{x}_2(t) &= 3e^{-2}x_2(t-2) - 5x_2^2(t) - x_2(t)y_2(t) - 2x_2^2(t)y_2^2(t), \\ \dot{y}_1(t) &= 2y_2(t) - y_1(t) - 2e^{-1}y_2(t-1), \\ \dot{y}_2(t) &= 2e^{-1}y_2(t-1) - 4y_2^2(t) - 2x_2(t)y_2(t). \end{aligned} \tag{5.1}$$

In this case, corresponding to system (1.6)

$$\begin{aligned} &\alpha_1 = 3, \quad \gamma_1 = 1, \quad \beta_1 = 5, \quad c_1 = 1, \quad \rho = 2, \\ &\alpha_2 = 2, \quad \gamma_2 = 1, \quad \beta_2 = 4, \quad c_2 = 2, \quad \tau_1 = 2, \quad \tau_2 = 1. \end{aligned}$$

By simple computation, one can see that

$$\frac{c_1}{\beta_2}=\frac{1}{4},\quad \frac{\beta_1}{c_2}=\frac{5}{2},\quad \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}}\approx 0.5518,\quad \frac{3\beta_2(\beta_1\beta_2-c_1c_2)e^{2\gamma_2\tau_2}}{\alpha_2^2}\approx 399.0090.$$

Clearly, condition (H_1) is satisfied. From Corollary 3.1, system (5.1) is globally attractive. Numeric simulation (Figure 2) supports this findings.

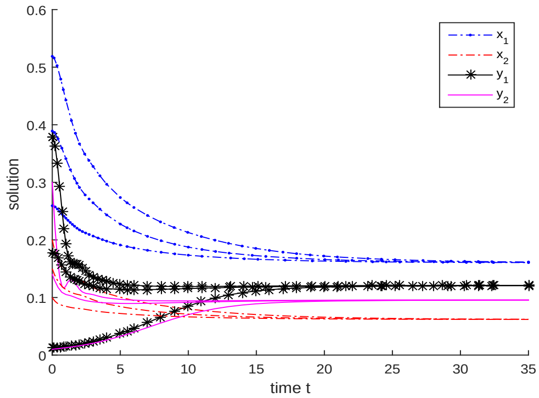

Figure 2: Dynamic behaviors of system (5.1) with initial values $(x_1(\theta),x_2(\theta),y_1(\theta),y_2(\theta))$ $=(0.5188,0.2,0.0126,0.01)$, $(0.2594,0.1,0.3793,0.3)$ and $(0.3891,0.15,0.1770,0.14)$, $\theta\in(-2,0]$, respectively.

Example 5.2 Now consider Example 1.1, in this case, one can easily check that

$$\frac{c_1}{\beta_2}=1,\quad \frac{\beta_1}{c_2}=2,\quad \frac{\alpha_1 e^{-\gamma_1\tau_1}}{\alpha_2 e^{-\gamma_2\tau_2}}\approx 0.5518.$$

Clearly, conditions (H_0') and (H_2) are satisfied. From Theorem 4.1, species 1 will be driven to extinction. The example illustrates that (H_0') is not sufficient condition which guarantee the permanence of system (1.6).

6 Conclusion

In this paper, we consider an autonomous stage-structured competitive systems with toxic effect. Theorem 3.1 shows that if the system without toxic substance is globally attractive, and if the rate of toxic substance is restrict to some range such that condition (H_1) holds, then the toxic substance has influence on the the stability property of the unique interior equilibrium of system (1.6). Theorem 4.1 shows that although the condition (H_0') holds, the first species still be driven to extinction if additional condition (H_2) holds.

One of the interesting issue is to find out the dynamic behaviors of system (3.1) under the assumption $\rho>\frac{3\beta_2(\beta_1\beta_2-c_1c_2)e^{2\gamma_2\tau_2}}{\alpha_2^2}$. We will leave this for future investigation.

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